APPLIED STATICS

BY

T. R. LOUDON, B.A.Sc.

FIRST EDITION

PRINTED FOR THE ENGINEERING SOCIETY
OF THE
FACULTY OF APPLIED SCIENCE OF TORONTO UNIVERSITY
BY THE CANADIAN ENGINEER PRESS,
TORONTO, ONT.
1909

Entered according to Act of the Parliament of Canada, in the year one thousand nine hundred and nine, by THE MONETARY TIMES PRINTING CO. OF CANADA, LIMITED, at the Department of Agriculture.

PREFACE.

It is the intention, in the following pages, to provide an elementary text to be used in conjunction with the lectures on Statics as given in the Faculty of Applied Science and Engineering at Toronto University. Recognizing the fact that the student very often wastes a great deal of time in working out the steps that are omitted in most mathematical textbooks, the writer has endeavored to give each step in full, sometimes at the expense of conciseness.

The writer wishes to acknowledge his indebtedness to Professor C. H. C. Wright, upon whose course of lectures the following work is based; and to Professor W. J. Loudon for many valuable suggestions.

THOS. R. LOUDON.

Toronto, September 25th, 1909.

CONTENTS

Chapter | Page

I. Mechanics 5

II. Statics 7

III. Graphical Representations—the Resultant—Components 9

IV. Moments—Analytical Conditions of Equilibrium—Couples 22

V. Determination of Unknown Forces—Tension and Compression — Stress in Truss Members 31

VI. The Funicular Polygon—Second Graphical Condition of Equilibrium 42

VII. The Method of Sections 49

VIII. The Beam 53

IX. Vertical Shearing Force and Bending Moment for Stationary Loads on a Beam 59

X. V.S.F. and B.M. for Moving Loads on a Beam 74

XI. Pulley Systems 78

XII. Friction 87

XIII. Wind Pressure 94

APPLIED STATICS

CHAPTER I.

MECHANICS.

In its general interpretation, the subject of Mechanics treats of the Laws of Motion, of the relations that exist between matter and motion, and of the laws of equilibrium of what are termed **forces.**

Force, although popularly regarded as a definite, concrete idea, is the same sort of thing as velocity or acceleration, and is measured by the rate at which a mass changes its velocity.

In Engineering Mechanics, however, it may be considered as a definite quantity, capable of being measured, and represented by a number, the unit of which is the weight of one pound avoirdupois.

We speak, therefore, in popular phraseology of a force of ten pounds, or of a force **acting on a body**, or of forces **acting in certain directions,** just as if a force were a concrete thing, these being merely convenient, although scientifically misleading, modes of expression, which have been so ingrained into the whole subject of Mechanics that it would require a vast amount of circumlocution to avoid their use.

The following subdivision of Mechanics is generally used:—

1. Kinematics, which treats only of the geometry of motion.

2. Dynamics, which treats of the relations between matter and motion.

3. Statics, which deals with the laws of equilibrium of forces.

Although the subject of Statics alone will be considered here, yet the student should bear in mind that the second section, that of Dynamics, treats of Newton's Laws of Motion, which are given here because of their importance, especially the Third Law, the applications of which are continually occurring in statical problems.

Newton's Laws of Motion.

I. A body continues in a state of rest or uniform motion in a straight line till disturbed from such a state by an impressed force.

II. Change of motion is proportional to the impressed force, and takes place in the direction of the impressed force.

III. **Action and reaction are equal and opposite.**

CHAPTER II.

STATICS.

Taking the definition of the subject of Statics as given in the last chapter, it is seen that the consideration of motion does not enter into statical problems. It may be said, however, that although the analysis of motion produced or the tendency to produce motion does not constitute a part of the subject of Statics, it is often very convenient to examine the motion, or tendency to produce motion, in order that the properties of a given force or set of forces may be intelligently obtained.

The only forces considered in this treatise are coplanar forces, or forces acting in one plane.

It is advisable, before attempting the discussion of statical problems, that the student familiarize himself with the meanings of the terms given below:—

Magnitude—A property which admits of being measured. Quantities having this property may be relatively compared.

Direction—The common idea of direction is in reality compounded of two ideas, viz.: The line along which a certain manifestation may take place, or along which one body lies relative to another body; and which **way** along the line the manifestation takes place, or the one body lies relative to the other body. Direction, in the following pages, will be understood to mean merely the line along which action is manifested; concisely, **line of action.**

Sense—One of two **ways** in which a magnitude may be described or generated along a direction.

This distinction between Direction and Sense is more clearly seen by examples:—

A body is said to move vertically upward; i.e., the body moves in the vertical direction with an upward sense.

A building is said to lie east of another building; i.e., the first building lies in the direction of a parallel of latitude relative to the other building, and in an easterly sense.

A Vector.—A line conceived to have fixed length and direction in space, but whose position is not limited; i.e., the line may be drawn anywhere.

A Vector Quantity.—Any quantity having the properties, Magnitude, Direction, and Sense, is known as a vector quantity; e.g., Force, Velocity, Displacement, Acceleration. Such quantities may be graphically represented by Vectors.

A Rigid Body is one whose original form cannot be altered; i.e., the body cannot be compressed, extended, broken, or bent. Such a body, of course, exists merely in theory.

CHAPTER III.

GRAPHICAL REPRESENTATIONS—THE RESULTANT—COMPONENTS.

Graphical Representation of Force.

A Force, being a Vector Quantity, may be represented by a right line; for the direction of the line may represent the line of action of the force, the magnitude and sense being represented by first cutting off a portion of the line to some convenient scale of magnitude, and by then placing on this portion an arrow head pointed in the sense of action of the force.

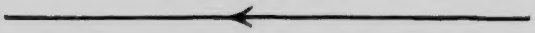


Fig. 1.

Fig. 1 represents a force of ten pounds magnitude acting in the horizontal direction, with sense to the left, as indicated by the arrow head. The scale of magnitudes in this case is $\frac{1}{4}$ inch = 1 pound.

Fig. 2 represents a set of forces acting at a point. In this diagram the magnitudes of the forces are represented by the lengths of the various lines. It is often more convenient, however, merely to represent the directions and senses accurately, and to place the numerical values of the magnitudes beside the lines representing the respective forces as in Fig. 3. Sometimes, when general conditions are merely wished in the form of a note, the directions as well as the magnitudes are only relatively represented, and the actual direction ratios, as well as the magnitudes, jotted down on the diagram as in Fig. 4.

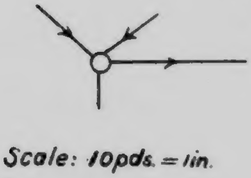


Fig.2.

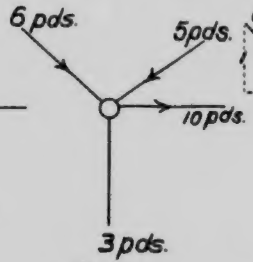


Fig.3.

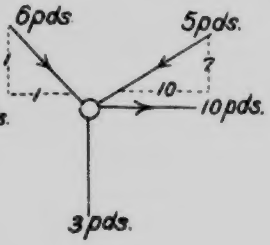


Fig.4

A diagram, such as either Fig. 2, Fig. 3, or Fig. 4, in which is given all the relative information of a set of forces, will be referred to as the **Statical Diagram** of the set of forces in question.

Bow's Notation.

In order that the graphical representation of any particular force may be intelligently and concisely referred to, a system of lettering known as Bow's Notation will be used throughout the work whenever possible. In this notation, the areas between the lines representing the forces are lettered, and a force is known by the letters to the areas on each side of the line representing that particular force.

Referring to the Statical Diagram (Fig. 5), the force of 50 pounds is known as either the force AB or BA, the force of 30 pounds as either the force BC or CB, and the force of 60 pounds as either the force CA or AC.

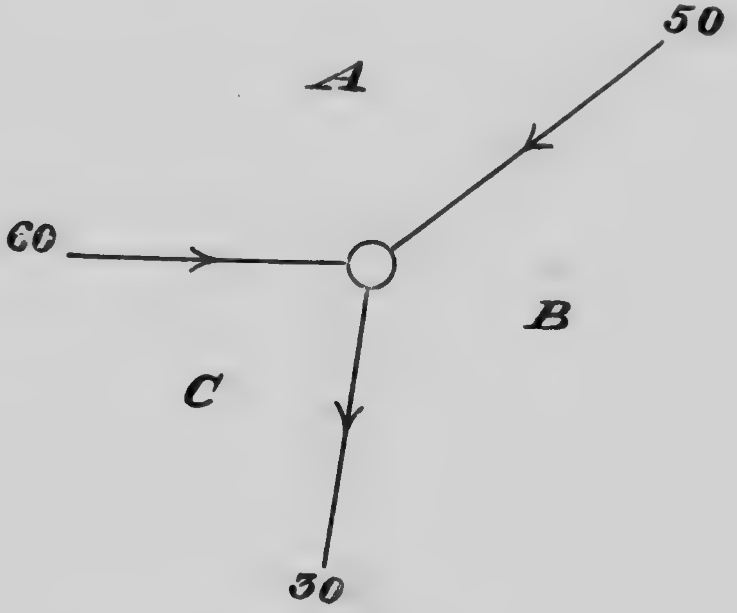


Fig. 5.

The Resultant Force is that one force which may replace and produce the same effect as a set of forces.

It is usual to refer to the Resultant Force merely as the **Resultant.**

It is evident that no resultant can be found for a set of forces in equilibrium. Because, if that could be done, there would be a single force (the resultant) acting on the body, and, therefore, motion would take place, which is impossible with equilibrium.

The Equilibriant or Balancing Force of a set of forces is that one force which, when acting with the set of forces, produces equilibrium.

The Equilibriant and Resultant of any given set of forces are of equal magnitude, and act in the same direction, but with opposite senses; or concisely, they are two equal and opposite forces. For, since the Resultant produces the same effect as the set of forces, it is manifest from definition that the Equilibriant would, when acting with the Resultant, produce equilibrium. Experimentally, it may be shown that the only possible manner in which two forces acting on the same body can give a state of equilibrium is that they be of equal magnitude and act in the same line with opposite senses.

Graphical Determination of the Magnitude, Direction, and Sense of the Resultant of a Set of Forces.

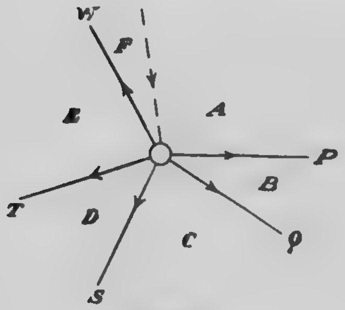


Fig. 6.

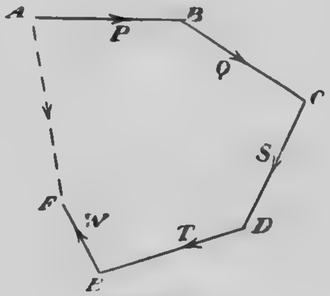


Fig. 6a.

Let there be given any set of forces, but, to make the problem clear, let us take a set of forces acting at a point. Fig. 6 is the Statical diagram of this set. Choose any initial point A as in Fig. 6a. From this point draw a line AB to represent **accurately** the force P in Fig. 6. From the point B, the line BC is drawn to represent the force Q (Fig. 6). In the same way the lines CD, DE, and EF are drawn to represent the forces S, T, and W, respectively. If now, the initial point A be joined to the final point F, then AF represents fully the Resultant of the forces P, Q, S, T, and W. **The proof of this statement is experimental.**

The accuracy of the representation o. the resultant in this last diagram depends entirely upon the care with which the relative directions and magnitudes of the forces are represented. If the directions of the forces relative to one another are represented accurately in the statical diagram, the simplest way of constructing the polygon is to draw the composing lines parallel to corresponding lines in the statical diagram.

The only restriction that is placed upon the construction of this polygon is that **the forces must be taken in such order that the sense marks in the polygon point continuously from the initial to the final point.** The sense mark of the resultant is always from the initial to the final point, and is, as it were, counter to the other sense marks.

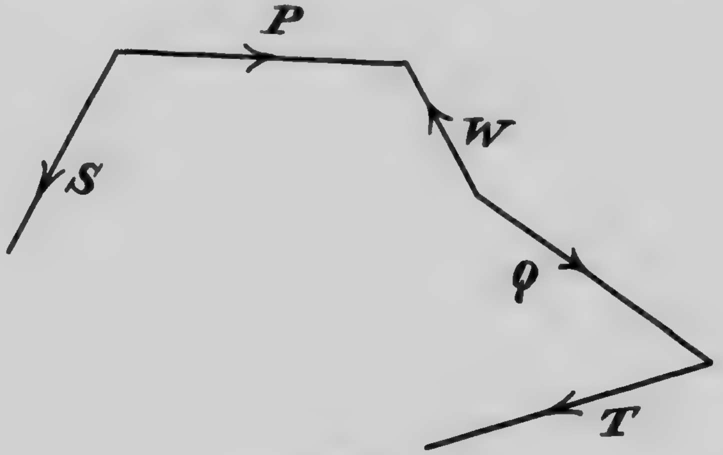


Fig. 7.

Fig. 7 shows an improper way of constructing the diagram. It will be noticed that the sense marks do not point continuously around the diagram from the initial to the final point. The resultant may, however, be found by a different arrangement of the representations than that shown in Fig. 6a. Fig. 8 shows a different diagram properly constructed with sense marks pointing continuously from the initial to final point. The resultant as found in this last case will have exactly the same magnitude, direction and sense as that found in Fig. 6a.

In the case where the forces do not act at a point, the procedure is exactly the same, the forces being chosen in such order that the sense marks on their representations point continuously from the initial to the final point of the diagram.

Diagrams such as Fig. 6a and Fig. 8 will be referred to in future as **Vector Polygons** or **Vector Diagrams,** and must not be confused with the Statical Diagram. The distinction lies in the fact that the Statical Diagram gives all information regarding the actual relatve positions of the forces as they lie in space, as well as the properties of the forces; whereas, only the magnitudes, directions, and senses of the forces need be known to construct the Vector Polygon.

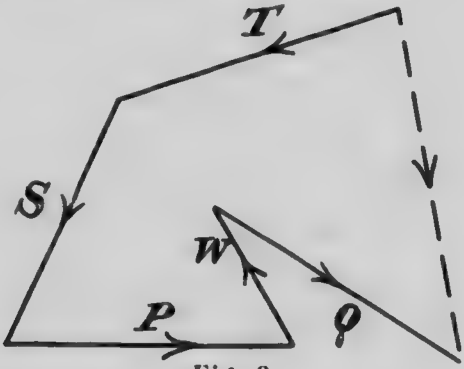


Fig. 8.

For a set of forces acting at a point, the resultant must act through this point in order to produce the same effect. Also, since the equilibriant acts in the same direction but in the opposite sense to the resultant, the equilibriant of a set of forces acting at a point must also act through the point. This gives a method of locating the resultant and equilibriant of a set of forces acting at a point, but for the general case, where the forces do not act at a point, the location of the resultant and equilibriant does not admit of such an easy solution, as will be seen later.

First Graphical Condition of Equilibrium.

It has been shown that a set of forces in equilibrium can have no resultant. Therefore, if the vector polygon be constructed for such a set of forces, the distance between the initial and final points, which represents the resultant, must be equal to zero; i.e., the initial and final points will coincide, or, in other words, the polygon closes. From this, then, **the first graphical condition of equilibrium is that the vector polygon must close.**

Components.

It has been pointed out that a set of forces may be replaced by a single force. Conversely, a force may be replaced by a set of two or more forces. The forces comprising such a set are known as **Components** of the replaced force, and, since the force is interchangeable with any set of its components, it follows that **a force is the resultant of a set of its components.** This may be more clearly seen from Fig. 9.

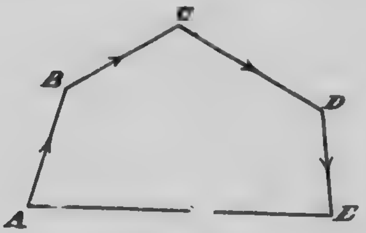


Fig. 9.

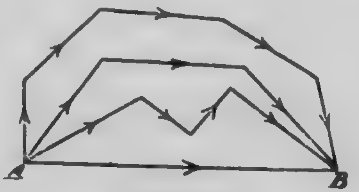


Fig. 10.

Let AE represent any force. AE is the resultant of AB, BC, CD, and DE; i.e., AE may replace AB, BC, CD, and DE. Conversely, the set AB, BC, CD, and DE may replace AE; i.e., they are a set of components. AE, the original force, is then the resultant of any of its component sets.

From this last statement it is seen that, if from the initial point of the representation of a force a number of lines be drawn continuously till the final point of the original representation is reached, these lines will represent a set of components. Fig. 10 shows a number of sets of components to the force AB. It is easily seen that an infinite number of such sets may be found.

To any force there may be found an infinite number of sets of components.

Resolved Parts.

In many of the problems of Statics, it is convenient to consider a force as replaced by a set of components, and in order to give definiteness to the problem, it is usual to replace the original force by two components whose directions are inclined at right angles to one another. Such a pair of components are known as **Resolved Parts,** and since a pair of resolved parts must always act perpendicularly to one another, the direction

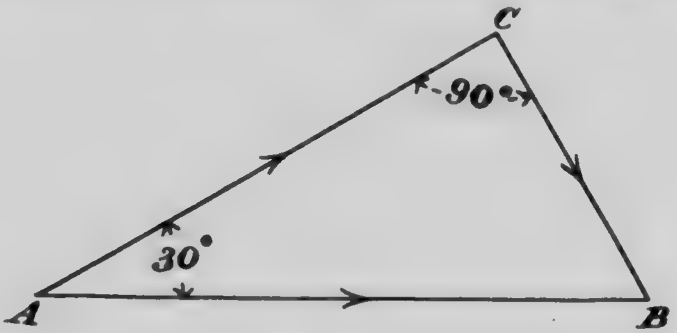


Fig. 11.

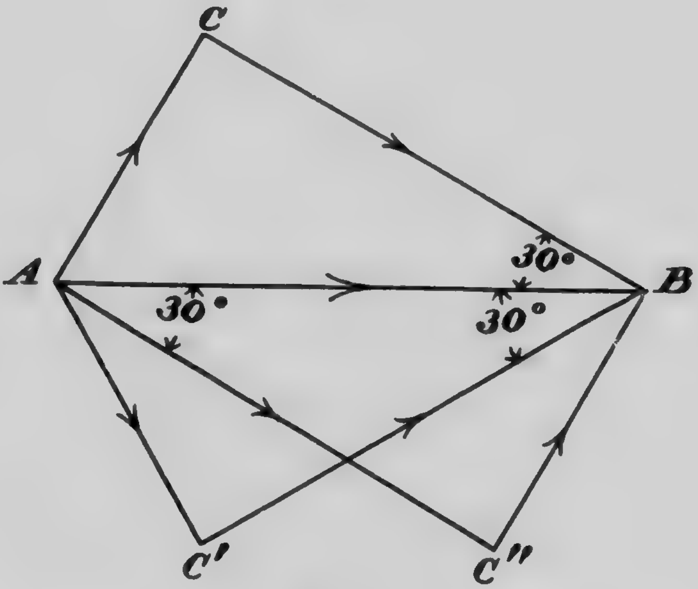


Fig. 12.

of one resolved part being given fixes the direction of the remaining resolved part. The operation of finding the value of a pair of resolved parts in any given directions that are perpendicular to one another, is known as **resolving** the original force in those directions.

Example:—

A force of 50 pounds acts horizontally to the right. Find the magnitude of a pair of resolved parts, one of which is inclined to the horizontal at thirty degrees.

Until the student has become familiar with the operation of resolving forces, it is advisable to draw a rough graphical representation of the force and the required resolved parts, and then, from the data, find by trigonometrical means the required values.

Let AB (Fig. 11) represent the given force. AC and CB evidently represent the required resolved parts, for AC makes an angle of 30° with the horizontal, and CB is perpendicular to AC. Then

$$AC = AB \cos 30^\circ = 50 \times \frac{\sqrt{3}}{2} = 25\sqrt{3}.$$

$$CB = AB \sin 30^\circ = 50 \times \tfrac{1}{2}. = 25,$$

which gives the magnitudes of the resolved parts in terms of the original force.

There are three other correct solutions to the problem which are shown in Fig. 12.

AC, CB; AC′, C′B; AC″, C″B are resolved parts which fill the requirements of the problem.

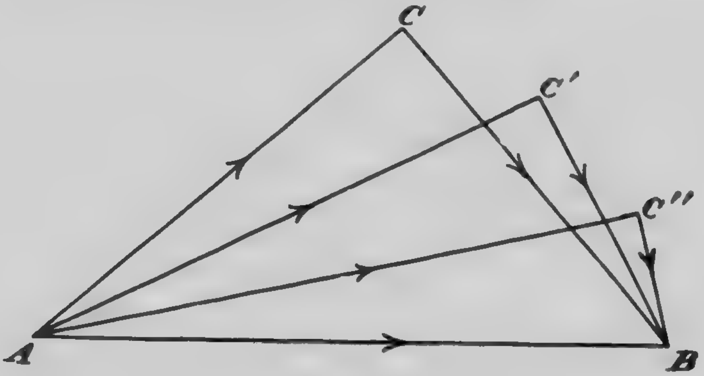


Fig. 13.

Resolved Parts Perpendicular and Parallel to a Given Force.

AC, CB; AC′, C′B, and AC″, C″B (Fig. 13) represent sets of resolved parts of AB. The directions of AC, AC′, and AC″ are respectively nearer to coincidence with the direction of AB. The directions of CB, C′B, and C″B, on the other hand, are respectively more nearly perpendicular to AB. Also, AC″ is nearer the value of AB than AC, and C″B is closer to the value zero than CB. Consequently, if AB be resolved in directions coincident with and perpendicular to itself, the magnitudes of the resolved parts would be respectively equal to AB and zero. In general, then, **the magnitude of a resolved part perpendicular to a given force is equal to zero,** and **the corresponding resolved part, whose direction must be coincident with the given force, is equal to the given force.**

Horizontal and Vertical Resolved Parts.

In engineering statical problems, it is customary, unless otherwise stated, to confine attention to resolved parts in the horizontal and vertical directions, and in order that these may be concisely referred to, the horizontal resolved part is called the X, and the vertical resolved part the Y of the force in question. Furthermore, as will be seen, since the solution of many problems depends upon the algebraic summation of the X's and Y's of a set of forces, a convention as to the positive and negative sign of X's and Y's must be agreed upon.

Horizontal resolved parts (X's) acting to the right and vertical resolved parts (Y's) acting upward shall be considered positive.

Magnitudes of the X and Y of any Force whose Inclination to the Horizontal is Given.

Let AB (Fig. 14) represent a force of magnitude P. If the inclination of the line of action of the force to the horizontal be the angle α as shown, then AC and CB represent X_P and Y_P respectively.

$$X_P = AC = AB \cos \alpha = P \cos \alpha.$$
$$Y_P = CB = AB \sin \alpha = P \sin \alpha.$$

If P act to the left as represented by AB′, and the angle of inclination still remain α, X_P is negative, but its magnitude is still given by $P \cos \alpha$. The Y_P is positive and still of magnitude $P \sin \alpha$.

It is seen, then, that the magnitude of the X of any force, irrespective of its sign, is given by the **product of the magnitude of the force and the cosine of the**

angle of inclination to the horizontal. The magnitude of the Y of any force is given by the **product of the magnitude of the force and the sine of the angle of inclination to the horizontal.**

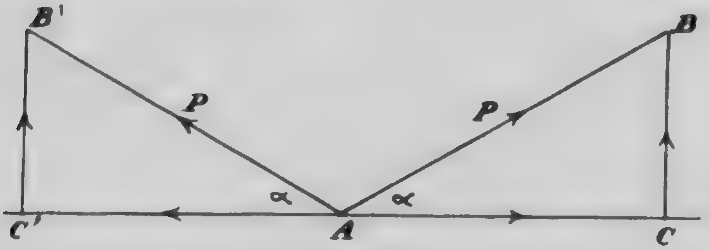


Fig. 14.

The X and Y of Horizontal and Vertical Forces.

If P be a horizontal force, then $\alpha = 0$, and

$$X_P = P\cos\alpha = P\cos 0^\circ = P.$$

$$Y_P = P\sin\alpha = P\sin 0^\circ = 0.$$

If P be a vertical force, then $\alpha = 90^\circ$, and

$$X_P = P\cos\alpha = P\cos 90^\circ = 0.$$

$$Y_P = P\sin\alpha = P\sin 90^\circ = P.$$

That is: **The horizontal resolved part of a horizontal force and the vertical resolved part of a vertical force are both equal to the force itself.**

The vertical resolved part of a horizontal force and the horizontal resolved part of a vertical force are both equal to zero.

CHAPTER IV.

MOMENTS—ANALYTICAL CONDITIONS OF EQUILIBRIUM—COUPLES.

If a body be pinned at some one point and acted on by any force not passing through this point, there will be produced a rotation, or a tendency to rotate. If a new point be chosen, or if a new force be introduced, this tendency to rotate will also change.

It may be shown that the turning effect varies as the product of the magnitude of the force and the perpendicular distance of the force from the point. To this product is given the name of the **Moment** of the force.

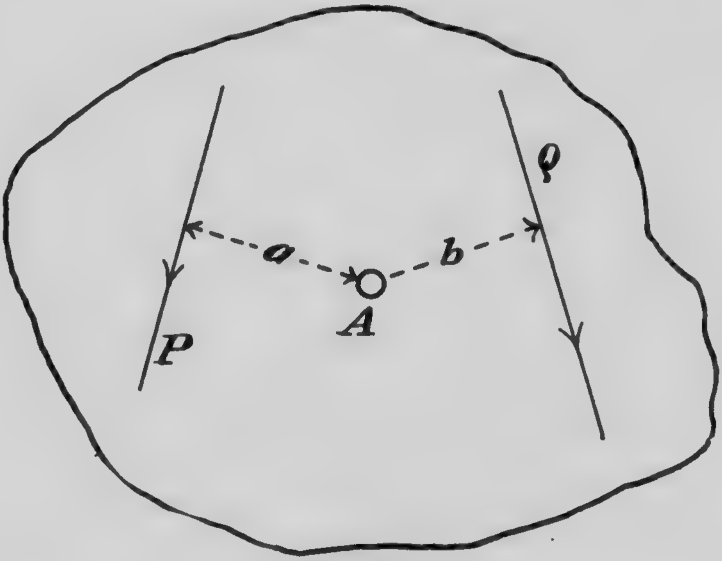


Fig. 15.

Definition :—

The moment of a force about a given point is the product of the magnitude of the force and the perpendicular distance of the point from the line of action of the force.

Since it is often necessary to find the algebraic sum of the moments of a set of forces about some one point, a convention as to the positive and negative sign of moments has to be agreed upon. In all the following problems, **forces which produce or tend to produce**

rotation of the body acted upon about the point in question in the sense of the motion of the hands of a clock have positive moments.

It may be said here that the other sense of rotation is very often used as the positive. It really does not matter which convention is used **as long as one convention is adhered to throughout the same problem.**

As an illustration, let P and Q (Fig. 15) represent two forces acting on a body, and let A be any point in the body, a and b being the perpendicular distances of P and Q, respectively, from the point A.

Imagining the body to be pinned at A, the force P would cause or tend to cause a rotation of the body in the counter clockwise sense of motion. The moment of P about A is, therefore, negative, and may be concisely written—

$$M_P = -P.a.$$

Under the same conditions, Q would cause or tend to cause a rotation of the body in the same sense as the motion of the hands of a clock. Q, in consequence, is said to have a positive moment, so that—

$$M_Q = +Q.b.$$

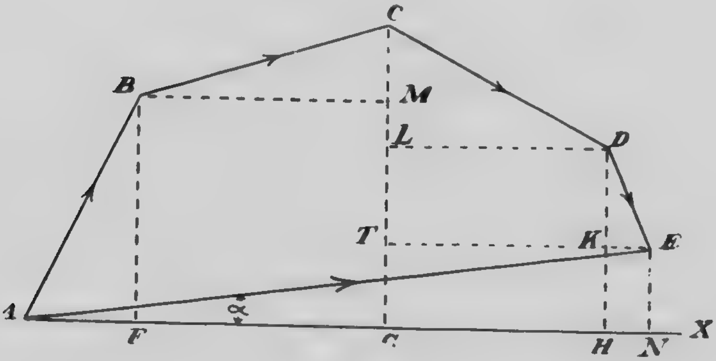


Fig. 16.

Analytical Determination of the Resultant.

To show that $\Sigma X = X_R$ and $\Sigma Y = Y_R$.

Let ABCDE (Fig. 16) be the Vector Polygon of a set of forces, the resultant of which it is required to find analytically.

Through A draw AX to represent the horizontal direction and from B, C, D, and E drop perpendiculars to AX, intersecting AX at F, G, H, and X, respectively. Through B, D, and E draw BM, DL, and KE parallel to AX, intersecting CG and DH at M, L, and K, respectively.

The horizontal and vertical resolved parts of the forces represented by AB, BC, CD, and DE are represented by the following lines:—

$X_{AB} = AF.$ $\quad Y_{AB} = FB = GM.$
$X_{BC} = BM = FG.$ $\quad Y_{BC} =$ [illegible]
$X_{CD} = LD = GH.$ $\quad Y_{CD} = -LC.$
$X_{DE} = KE = HX.$ $\quad Y_{DE} = -KD = -TL.$

Denoting the algebraic sum of the horizontal resolved parts by ΣX and the algebraic sum of the vertical resolved parts by ΣY—

$$\begin{aligned}\Sigma X &= X_{AB} + X_{BC} + X_{CD} + X_{DE}.\\ &= AF + FG + GH + HX.\\ &= AX.\end{aligned}$$

But $AX = X_{AE} =$ horizontal resolved part of the resultant; i.e., $\Sigma X = X_R$.

$$\begin{aligned}\Sigma Y &= Y_{AB} + Y_{BC} + Y_{CD} + Y_{DE}.\\ &= GM + MC - LC - TL.\\ &= GT = XE.\end{aligned}$$

But $XE = Y_{AE} =$ vertical resolved part of the resultant; i.e., $\Sigma Y = Y_R$.

If the inclination of the line of action of the resultant AE to the horizontal be denoted by α, then—

$$\tan \alpha = \frac{XE}{AX} = \frac{\Sigma Y}{\Sigma X}$$

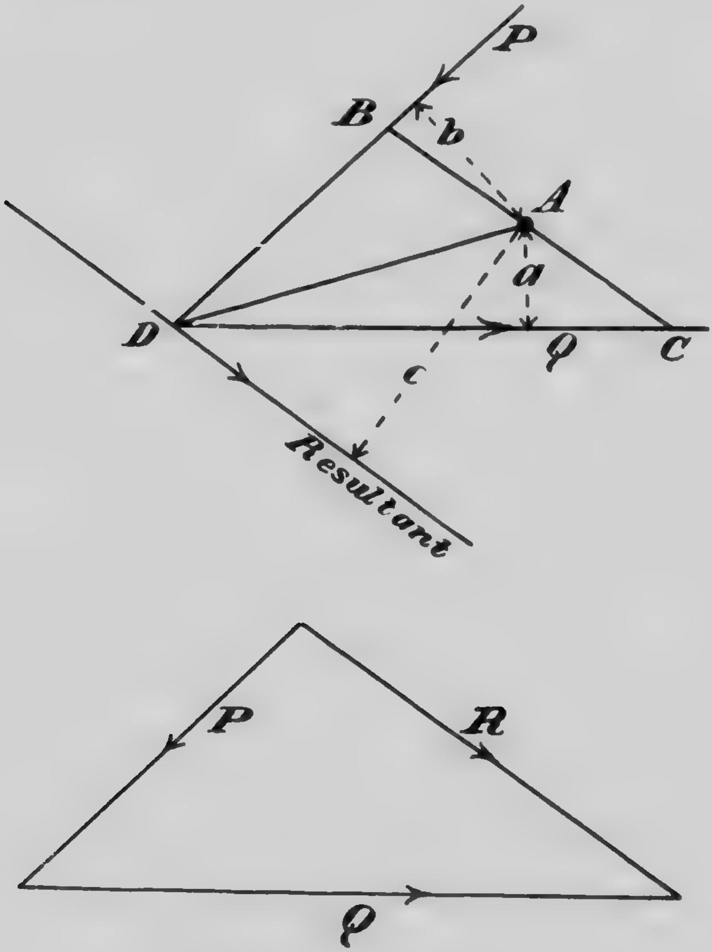
$$\text{and } R = AE = \sqrt{AX^2 + XE^2} = \sqrt{\Sigma X^2 + \Sigma Y^2}.$$

To show that $\Sigma M = M_R$.

Consider any two forces, P, Q, as in Fig. 17.

Determine, by means of the vector polygon, the resultant R of P and Q. This resultant will act at D, the intersection of the lines of action of P and Q.

Choose any point A about which to take moments, join A to D and through A draw a line parallel to the line of action of the resultant R, intersecting P and Q at B and C, respectively. The perpendicular distances of A from the lines of action of P, Q, and R are b, a, and c, respectively.



Vector Polygon

Fig. 17.

Because the lines BD, DC, and BC are situated relatively to one another, as are the directions of P, Q, and R, the triangle BDC may be taken as a vector polygon of the forces P, Q, and their resultant R, so that—

$$P = BD;\ Q = DC;\ R = BC.$$

Taking moments about A—

$$M_P = -P.b = -BD.b = -2\ \Delta\ ABD.$$
$$M_Q = -Q.a = -DC.a = -2\ \Delta\ ADC.$$
$$M_R = -R.c = -BC.c = -2\ \Delta\ BDC.$$

Denoting the algebraic sum of the moments of the forces P and Q by ΣM—

$$\Sigma M = M_P + M_Q.$$
$$-2\ \Delta\ ABD - 2\ \Delta\ ADC.$$
$$-2\ \Delta\ BDC.$$
$$\text{But } -2\ \Delta\ BDC = M_R.$$
$$\text{Therefore, } \Sigma M = M_R.$$

That is, the algebraic sum of the moments of two forces about any point is equal to the moment of their resultant about the same point.

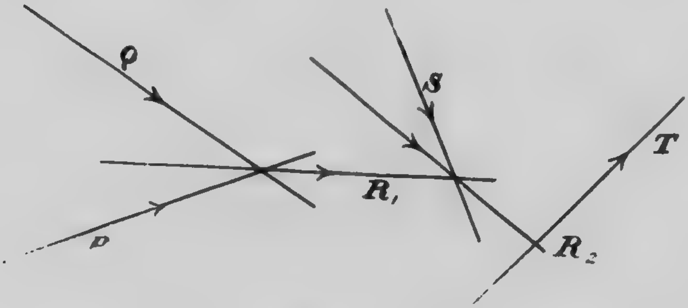


Fig. 18.

That the proof of the last proposition will apply to a case where the set of forces is more than two in number may be seen by considering a set of forces such as P, Q, S, and T represented in Fig. 18. P and Q may be replaced by their resultant R_1, so that R_1, S, and T are equivalent to the original set of forces. R_1 and S may be replaced by their resultant R_2, leaving R_2 and T as equivalent to the original four forces; that is, any number of forces comprising a set may be finally reduced to an equivalent pair of forces by such a system of combining and substituting resultants, and to the final pair so found may be applied the preceding proof. **Therefore, $\Sigma M = M_R$ for any set of forces.**

In Fig. 17, the point A might have been taken in several other positions relative to P, Q, and R. If the point be taken in the line of action of either P or Q, the moment of either P or Q, respectively, would be zero. If the point be taken in the line of action of R, it is evident that a new construction must be devised. In this case, draw through the point a line parallel to either the force P or Q, producing it to intersect the line of action of the remaining force. The triangle so formed may be taken as a vector polygon as before and the same proof applied, the moment of R being zero.

Tabulating these analytical results:

$$\Sigma X = X_R. \qquad \Sigma Y = Y_R. \qquad \Sigma M = M_R. \qquad \tan \alpha = \frac{\Sigma Y}{\Sigma X} \qquad R = \sqrt{\Sigma X^2 + \Sigma Y^2}.$$

From these, the magnitude, direction, sense, and location of the resultant of any set of forces may be fully determined.

Analytical Conditions of Equilibrium.

It has previously been shown that a set of forces in equilibrium can have no resultant. From this it follows that since the value of the resultant is zero, the X, Y, and M of R must be equal to zero, and consequently:-

$$\Sigma X = 0.$$
$$\Sigma Y = 0.$$
$$\Sigma M = 0.$$

are the analytical conditions of equilibrium.

Couples.

Definition: A couple is a pair of equal forces which act in parallel directions with opposite senses.

Fig. 19 represents a couple. The two forces are of magnitude P, and are a distance *a* apart.

If the vector polygon be drawn for these two forces it must necessarily close.

Let α be the angle that the directions of the forces make with the horizontal.

Applying the analytical conditions ΣX, ΣY, and ΣM—

$$\Sigma X = P \cos \alpha - P \cos \alpha = 0.$$
$$\Sigma Y = P \sin \alpha - P \sin \alpha = 0.$$

Taking moments about any point A distant x from one of the forces—

$$\Sigma M = P\,(a + x) - P\,x = P.a.$$

If A be taken in any other position, it is found that the result is always the same, and since P and a are both constants—

$$\Sigma M = P.a = \text{Constant}.$$

Summary:—

For a couple, then, the vector polygon must close.

$$\Sigma X = 0.$$
$$\Sigma Y = 0.$$
$$\Sigma M = P.a = \text{Constant}.$$

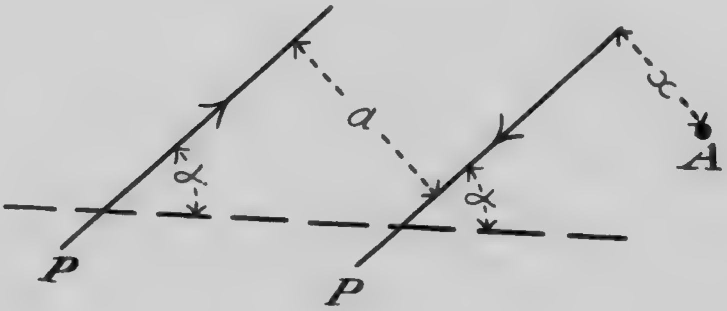


Fig. 19.

CHAPTER V.

DETERMINATION OF UNKNOWN FORCES—TENSION AND COMPRESSION—STRESS IN TRUSS MEMBERS.

Let Fig. 20 represent a plank resting on an abutment A such that 10 feet and 30 feet of the total length overhang to the left and right, respectively. If 100 pounds be placed on the left hand end of the beam, what must be the magnitude of the force P exerted at the other end in order to preserve equilibrium, and what is the magnitude of the abutment reaction A?

Consider the beam as a rigid body. The forces acting on the body are: The abutment reaction, acting vertically upward with a magnitude of A pounds, and the two forces of magnitude, 100 and P pounds, acting vertically downward, and these three forces are in equilibrium; therefore, $\Sigma X = 0$, $\Sigma Y = 0$, and $\Sigma M = 0$.

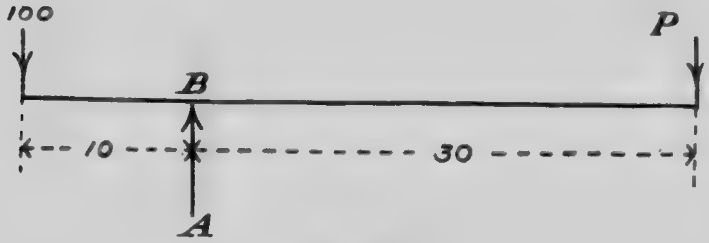


Fig. 20.

Apply these analytical conditions of equilibrium:—

$\Sigma X = X_{100} + X_A + X_P = 0.$

$0 + 0 + 0 = 0$ since all the forces act vertically.

$\Sigma Y = Y_{100} + Y_A + Y_P = 0.$

$-100 + A - P = 0.$

$A - P = 100$ .... (1)

Take moments about any point in the abutment, say, B. (The convention as to positive and negative moments must be kept in mind.)

$$\Sigma M = M_{100} + M_A + M_P = 0.$$

$$100 \times 10 + 0 + P.30 = 0.$$

$$P = \frac{1000}{30} \quad \ldots\ldots \; (2)$$

Substituting this value of P found in (2) into (1).

$$A - \frac{100}{3} = 100.$$

$$A = 100 + \frac{100}{3}$$

It will be noticed that the equation $\Sigma X = 0$ gave absolutely no information, and also that if P had been the only quantity required, the equation $\Sigma M = 0$ would have given it; that is, it is not always necessary to use all three equations to find an unknown quantity. As to which one should be used for a given problem or whether two or all three have to be used, experience alone in working problems will give the student the ability to determine this at once by inspection. No mistake can be made, however, by using all three equations.

If a set of forces are in equilibrium, it does not signify about what point moments are taken; but it is best to choose a point in the line of action of one of the unknown forces, for by so doing the moment of that force becomes equal to zero, thus eliminating an unknown quantity and making it possible to solve the problem with one equation if there are only two unknown forces in question.

Abutment Reactions.

Fig. 21 represents a truss supported on two abutments; required to find the value of the abutment reactions AE and DC.

Let the unknown values of AE and DC be P and Q, respectively.

The truss is acted upon by a set of outside forces which are in equilibrium; therefore, for this set of forces, $\Sigma X = 0$, $\Sigma Y = 0$, and $\Sigma M = 0$.

Take moments about any point in the line of action of either abutment reaction, say, DC.

$$\Sigma M = M_{AE} + M_{AB} + M_{BC} + M_{CD} + M_{DE} = 0.$$
$$P.40 - 100.30 - 100.10 + Q.0 - 100.20 = 0.$$
$$P.40 - 100\ (30 + 10 + 20) = 0.$$
$$P = 150$$
$$\Sigma Y = Y_{AE} + Y_{AB} + Y_{BC} + Y_{CD} + Y_{D}$$
$$150 - 100 - 100 + Q - 10$$
$$Q = 150$$

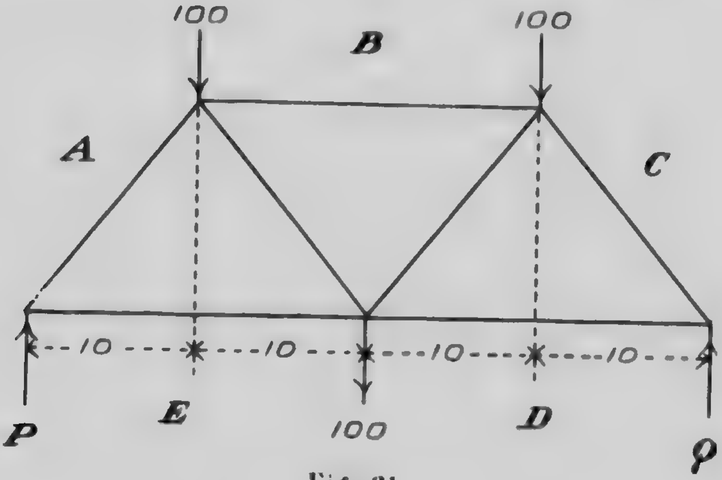


Fig. 21.

The use of the last equation is, in a sense, superfluous, for since the abutments support all the load, and if one abutment be found to support 150 pounds, then the other must support the remainder of the load. It is well for the student, however, to see that this reasoning is really based upon the condition $\Sigma Y = 0$.

Tension and Compression.

If a steel bar AB (Fig. 22) rest against a wall and a resultant force P be exerted at B against the bar, the bar is said to be **in compression.**

Keeping in mind Newton's third law of motion, that action and reaction are equal and opposite, it is evident that at B there must be a force exerted contrary and equal to P. This is indicated by D. If any section such as C be examined, it also follows that the right hand particles must be exerting a force equal to P, which is resisted by the left hand particles, as indicated by E and F. Arriving at the end A, there is exerted a push against the wall of magnitude P as represented by G.

If a steel bar AB (Fig. 23) be affixed to an abutment and a resultant force P′ be exerted so as to tend to pull the bar away from the abutment, the bar is said to be **in tension.**

At B there must be exerted an equal and opposite force to P as indicated by D, and at the end A the bar exerts a pull on the abutment as shown by G. If any section C be considered, it follows that the material tends to tear apart; that is, the forces acting at the section must be such as represented by E and F.

In both these cases the bars are said to be stressed, stress being a resistance to change of shape, and must not be confused with the term strain, which refers to the change in dimension due to stress.

In general, **If at any section of a body it is found that the resultant of the forces acting at the section acts against the section, the body is said to be in compression; or, if this resultant is found to be acting away from the section, the body is in tension.**

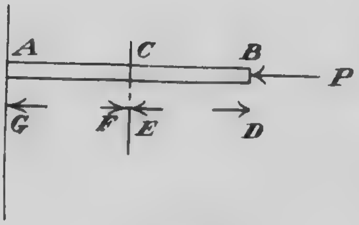


Fig. 22.

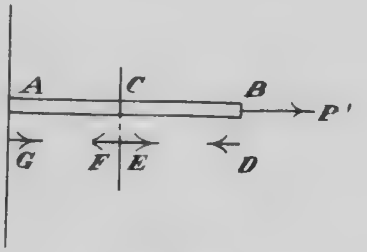


Fig. 23.

The Stress in Truss Members.

The following problem indicates the procedure in determining stresses in the members of a framed structure. The truss chosen to illustrate this is known as a cantilever, the principle of which is applied in bridge building the world over. The example given is necessarily a very simple case, but, while difficulties may arise in other problems relating to stress in truss members, they are found to be due to lack of knowledge as to where to begin the problem rather than to any change in the principle involved. Experience in working problems is the only remedy for this last-named obstacle.

Graphical Solution.

Fig. 24 represents a cantilever supporting a load of 100 pounds.

Consider the forces acting at the point ABC. In order to fix one's ideas on the point in question, it is always advisable to draw a statical diagram for that point, placing on it all the known data.

Fig. 25 is the statical diagram for the point ABC. The forces acting at this point are in equilibrium, one being completely known, the other two, AB and CA, being only partially known; i.e., their directions are known. **Such forces as AB and CA will always be referred to as unknown forces.**

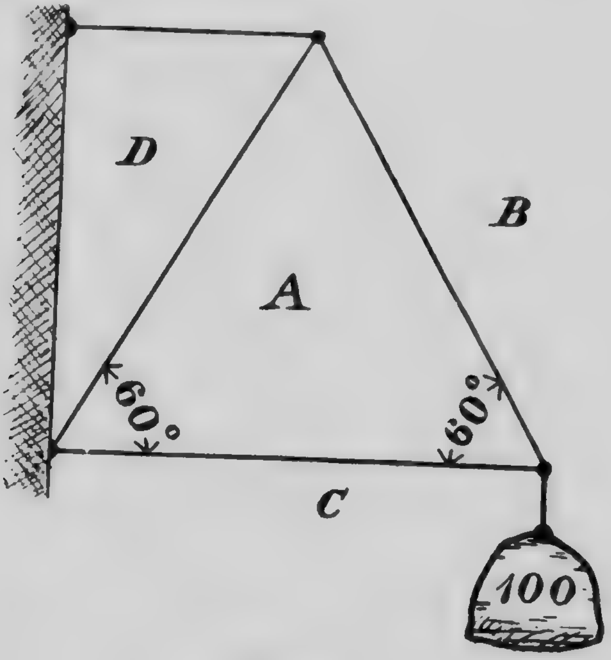


Fig. 24.

From any initial point B (Fig. 26) draw a line parallel to BC in Fig. 25 to represent the direction of the force BC. Cut off BC, in Fig. 26, to represent a magnitude of 100 pounds, placing the sense mark from B to C, thus totally representing the force BC.

From C (Fig. 26) draw a line parallel to CA in Fig. 25. Now, since the magnitude of the force CA is unknown, it is at present seemingly impossible to locate the point A on this new line, so that CA may represent the force CA. However, it is known that the forces BC, CA, and AB are in equilibrium, from which it follows that if the vector polygon be drawn for these forces it must close. So, then, from some point in the last line drawn from C a third line must be drawn parallel to AB (Fig. 25), such that it will pass through B (Fig. 26), thereby closing the vector polygon. This is best accomplished by drawing through B a line parallel to AB (Fig. 25), intersecting the other line last drawn, at A. It is evident from the previous discussion that CA and AB (Fig. 26) represent completely the hitherto unknown forces CA and AB, their senses being such as to give continuous sense from the initial point to the final point of the polygon as indicated.

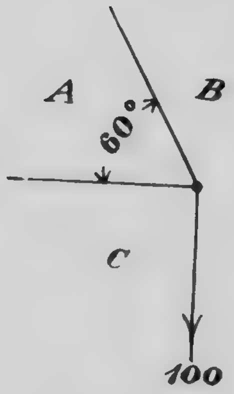


Fig. 25.

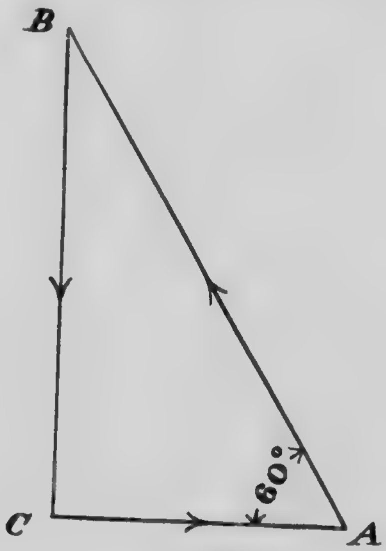


Fig. 26.

From the data on the statical diagram the triangle BCA is a 60° right angled triangle. It follows that since BC represents 100 pounds, CA and AB represent $\frac{100}{\sqrt{3}}$ and $\frac{200}{\sqrt{3}}$ pounds, respectively.

Placing the sense of CA and AB as found in the vector polygon, on the statical diagram, it is seen that

CA acts against the point and AB away from the point. The member CA is, therefore, in compression and AB in tension to the extent of $\frac{100}{\sqrt{3}}$ pounds and $\frac{200}{\sqrt{3}}$ pounds, respectively. (For the reasoning on tension and compression see page 34.)

Fig. 27 is the statical diagram for the point ABD. Since the member AB is in tension, there is a known force AB acting away from the point ABD as indicated. The two forces BD and DA are unknown.

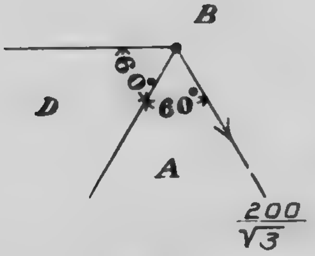


Fig. 27.

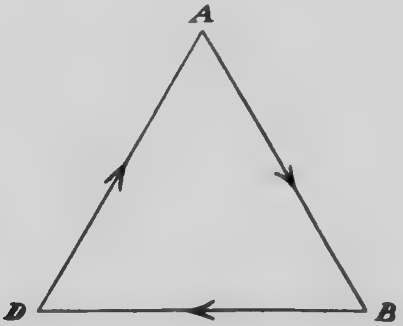


Fig. 28.

Referring to Fig. 28, AB represents the known force AB. From B is drawn a line to represent the direction of the unknown force BD, and from A a line is drawn to represent the direction of the second unknown force DA. These two lines intersect at D. BD and DA represent the unknown forces BD and DA, and because the triangle ABD is equilateral (from data), BD and DA must each be equal to AB; i.e., of $\frac{200}{\sqrt{3}}$ pounds magnitude. BD acts away from the point ABD and DA against the point. The members BD and DA are, therefore, in tension and compression, respectively.

The results of the preceding solution are given in Fig. 29, compression and tension members being shown by thickened and light lines, respectively.

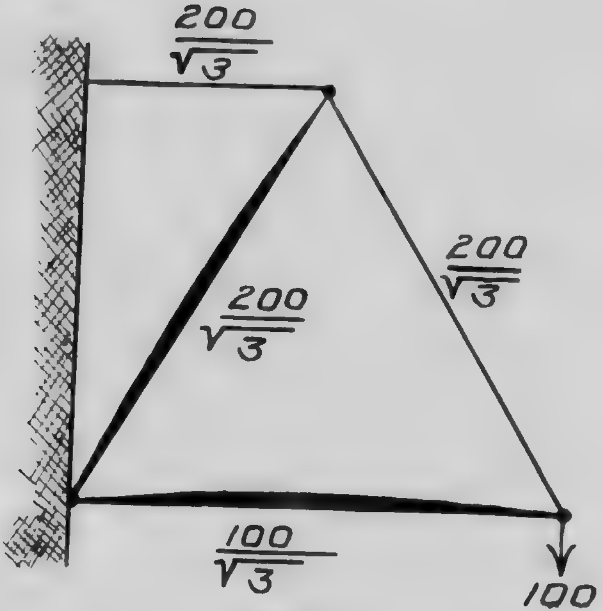


Fig. 29.

Analytical Solution.

Apply the analytical conditions of equilibrium to the points ABC and ABD.

Point ABC:—

$$\Sigma Y = Y_{CB} + Y_{BA} + Y_{AC} = 0.$$

$$-100 + BA \sin 60^\circ + 0 = 0.$$

$$BA = \frac{200}{\sqrt{3}}$$

This positive result shows that the Y_{BA} is positive. BA, therefore, acts away from the point ABC; i.e., the member BA is in tension $\frac{200}{\sqrt{3}}$ pounds.

$$\Sigma X = X_{CB} + X_{BA} + X_{AC} = 0.$$

$$0 - BA \cos 60^\circ + AC = 0.$$

$$-\frac{200}{\sqrt{3}} \cdot \frac{1}{2} + AC = 0.$$

$$AC = \frac{100}{\sqrt{3}}$$

From the positive result, the X_{AC} is positive: i.e., AC acts against the point. The member AC is, therefore, in compression $\frac{100}{\sqrt{3}}$ pounds.

Point ABD:—

$$\Sigma Y = Y_{AB} + Y_{BD} + Y_{DA} = 0.$$
$$-AB \sin 60^\circ + 0 + DA \sin 60^\circ = 0.$$
$$DA = AB$$
$$= \frac{200}{\sqrt{3}}$$

DA is in compression $\frac{200}{\sqrt{3}}$ pounds.

$$\Sigma X = X_{AB} + X_{BD} + X_{DA} = 0.$$
$$AB \cos 60^\circ + BD + DA \cos 60^\circ = 0.$$
$$BD = -(AB + DA) \cos 60^\circ.$$
$$= -\left(\frac{200}{\sqrt{3}} + \frac{200}{\sqrt{3}}\right). \frac{1}{2}.$$
$$= -\frac{200}{\sqrt{3}}.$$

BD is in tension $\frac{200}{\sqrt{3}}$ pounds.

In the last two equations the reasoning as to why the members are in compression and tension has been left out. The student is advised to work this out himself from the reasoning in the fromer equations for the point ABC.

These results, found analytically, check those found graphically. It will be noticed that in the analytical solution the use of the equation ΣM was not necessary. Cases will arise, however, where all three equations have to be used, and only experience can teach which equation should be first used to the greatest advantage in getting quick results. No mistake can be made in using any one of the equations first, but it will soon be seen that simultaneous equations may often be avoided.

CHAPTER VI.

THE FUNICULAR POLYGON—SECOND GRAPHICAL CONDITION OF EQUILIBRIUM.

Graphical Location of the Resultant of a Set of Forces.

It has been already shown that the resultant of a set of forces which act at a point also acts at the same point, thus making the problem of locating the resultant a very simple one for that particular case. There arise, however, numerous cases where the forces comprising a set do not act at a point, so that the previous method of locating the resultant does not apply. If the lines of action of the forces comprising the set intersect within reasonable limits, the resultant might be located as follows:—

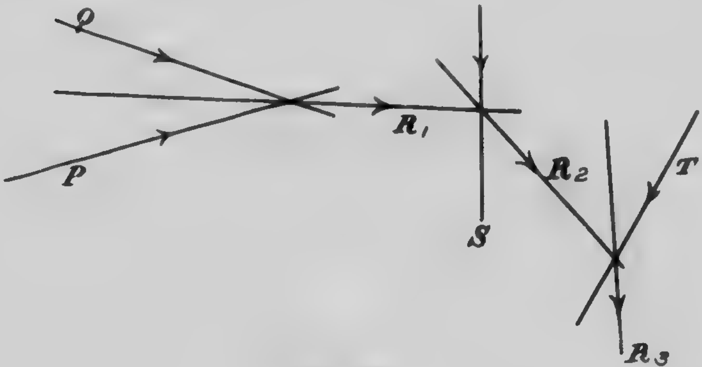


Fig. 30.

Let P, Q, S, and T (Fig. 30) represent a set of coplanar forces. Substitute for P and Q their resultant R1. R1, S, and T are then equivalent to the original set. Replace R1 and S by their resultant R2. R2 and T are also equivalent to the original set of forces. Substitute for R2 and T their resultant R3, which must then be the equivalent of the original forces P, Q, S, and T; i.e., R3 is the required resultant. Thus by a process of combination and elimination, the resultant of a set of forces may be located. When, however, some or all of the forces of a set have parallel or almost parallel lines of action, this method is impracticable.

Let P, Q, and S (Fig. 31) be any set of coplanar forces and ABCD their vector polygon, required to locate their resultant, which is represented by AD.

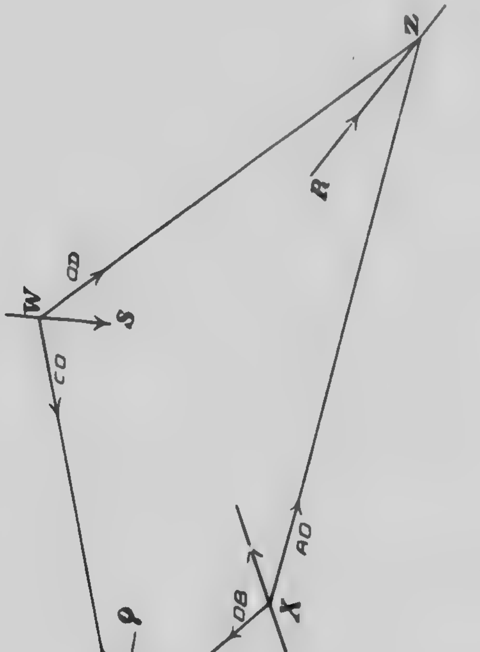


Fig. 31.

Take any point O inside or outside of the vector polygon and join it to all points which terminate the representations of the various forces; in this case, to A, B, C, and D. (It might have been a little clearer, in this particular case, to have said, join O to all corners of the vector polygon; but the above phraseology is more general and applies to the vector polygon of a set of parallel forces.)

Before proceeding with the argument, there are two facts that must be borne in mind throughout this proposition:—

(1) A force may be replaced by its components at any point on its line of action.

(2) The resultant of two forces acts through the intersection of the lines of action of the two forces.

Referring back to Fig. 31, AO and OB represent a pair of components of the force represented by AB; i.e., the force P. P may, therefore, be replaced by AO and OB at any point on its line of action. [See (1).] At any point X on the line of action of P introduce AO and OB. Produce the line of action of OB to intersect the line of action of Q at Y. Q may be replaced by the two components represented by BO and OC, and, since it may be replaced at any point on its line of action, choose the point Y. At Y introduce the components BO and OC. The line of action of BO will, of course, coincide with the line of action of the component OB of the force P. Produce the line of action of OC to intersect the line of action of S at W, and at W replace S by its two components represented by CO and OD.

The forces P, Q, and S have been replaced by their respective components, so that in place of the original forces there are now acting the forces AO, OB, BO, OC, CO, and OD. It is seen, however, that OB and BO are two forces of equal magnitude, acting with opposite senses in the same line of action. Their effect on the equilibrium of the body is, therefore, zero. In the same way, OC and CO counteract one another, thus leaving AO and OD, so that as far as actual effect is concerned **AO and OD are equivalent to the original forces.** But AO and OD are two components of AD, or conversely, AD is the resultant of the two forces, AO and OD; i.e., from (2), AO and OD may be replaced at the intersection of their lines of action (the point Z) by AD. AD, however, is the resultant of the original set of forces which it was required to locate.

The Funicular Polygon.

In Fig. 31 the polygon XYWZ, formed by the lines representing the directions of the replacing components is given the name of a **Funicular Polygon.** This polygon must not be confused in any way with the vector polygon, nor with the lines drawn from the point O to the vector polygon. This last mentioned point O is known as a "pole," which, together with the vector diagram and the lines to O, form a **Polar Diagram**.

The Resultant Couple.

There arise many cases of sets of forces where, if the vector polygon be drawn, it is found to close, and yet the forces are not in equilibrium. For instance, if in Fig. 32 the vector polygon ACB be drawn for the set of forces P, Q, and S as represented, it is found to close. But it is evident that such a set of forces would produce a rotation of the body: that is, there is not equilibrium, although the resultant force is zero.*

It will thus be seen that cases may arise when a set of forces acts upon a body, such that, although what we term the **resultant force** vanishes, there is yet a motion of rotation about some point in the body; and we speak of the forces as equivalent then to a **resultant couple.**

Draw the funicular polygon for P, Q, and S, and it will be found that the first and last components do not intersect one another, if their directions are produced, but that they have parallel lines of action.

Since these first and last components are of equal magnitude and act in the opposite sense, they form a couple, which is called the **Resultant Couple.**

Second Graphical Condition of Equilibrium.

If in the last proposition and in the one preceding it, the lines of action of the first and last components had been coincident, these two components would then counteract one another, so that the effect of all the replacing components would be zero, or, in other words, the original forces would be in equilibrium.

---

* Three forces to be in equilibrium must act at a point.

So, then, as a second graphical condition of equilibrium, **the first and last components of the funicular polygon must have their lines of action coincident.**

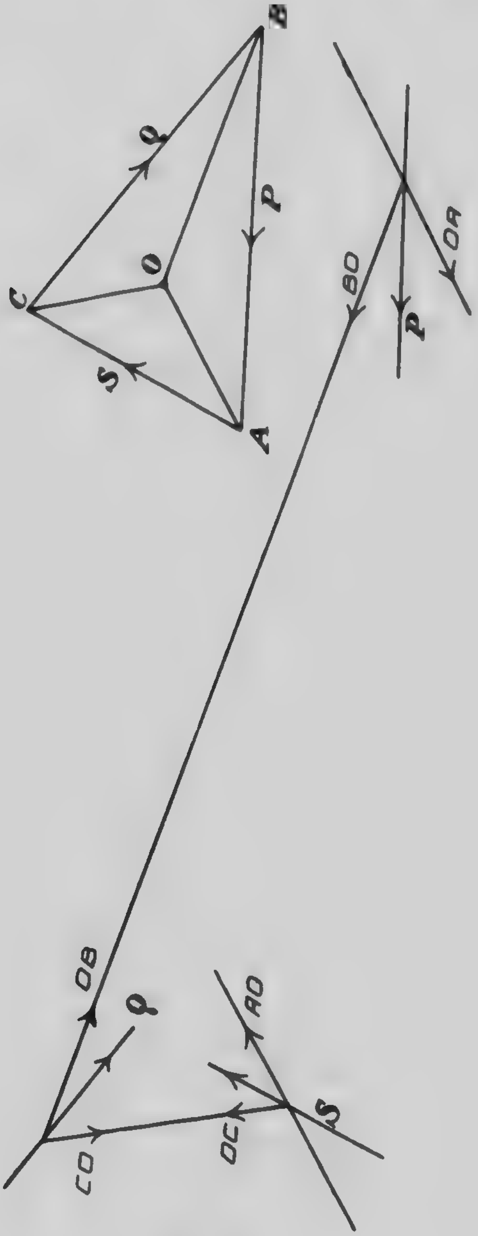


Fig. 32.

This is often more concisely stated by saying that **the funicular polygon must close**, keeping in mind what is meant by the word "close."

Graphical and Analytical Determination of the Resultant.

Graphical.	Analytical.
Vector Polygon gives the magnitude, direction, and sense of the resultant.	$\Sigma X = X_R$. $\Sigma Y = Y_R$. $R = \sqrt{\Sigma X^2 + \Sigma Y^2}$ $\tan \alpha = \dfrac{\Sigma Y}{\Sigma X}$
Funicular Polygon locates the line of action of the resultant.	$\Sigma M = M_R$.

Conditions of Equilibrium.

Graphical.	Analytical.
Vector Polygon must close.	$\Sigma X = 0$. $\Sigma Y = 0$.
Funicular Polygon must close.	$\Sigma M = 0$.

CHAPTER VII.

THE METHOD OF SECTIONS.

It is often necessary to find the stress in one or more members of a truss, irrespective of the stress in other members. To obviate the tedious operation of working from point to point till the stress in the required member is finally reached, what is known as the **Method of Sections** is used, whereby the stress, say, in EF (Fig. 33) may be arrived at immediately **from a consideration of the forces acting on the truss.**

Imagine the truss represented in Fig. 33 as cut into two sections at ZW. Consider the portion of the truss to the left of ZW. This considering the portion to the left is purely arbitrary. The same results would eventually be reached by considering the right-hand section, but in order to avoid confusion, the convention of considering the left-hand portion is agreed upon.

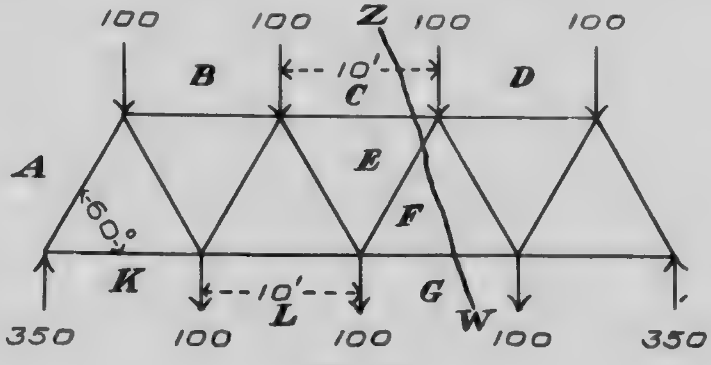


Fig. 33.

This portion of the truss to the left of ZW is held in equilibrium by certain forces **acting on it.** There is no difficulty in seeing that the left-hand abutment reaction AK, and the four loads AB, BC, KL, and LG, act on the body to left of ZW. **But that these forces alone could not preserve equilibrium** is easily seen. There must, then, be other forces acting on the body, and the only possible places where forces could act is where the portions of the members CE, EF, and FG, to the right of ZW, touch the portions to the left.

As far as the forces acting on the body are concerned, it does not matter whether the body be considered as a truss form or as a solid body having the boundary outline of the truss. It is generally less confusing, however, to choose the latter method, because it seems to concentrate attention on the extraneous forces.

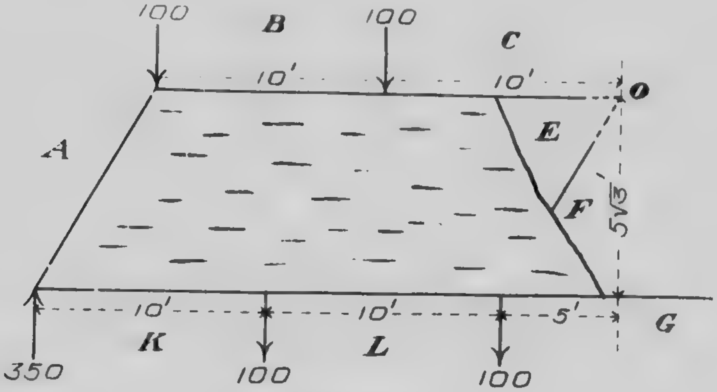


Fig. 34.

Fig. 34 represents the forces acting upon the portion of the truss to the left of ZW, considered as a solid rigid body. These forces are in equilibrium, and, since there are only three unknown forces, CE, EF, and FG, it is possible to solve for them by means of the three analytical equations of equilibrium, $\Sigma X = 0$, $\Sigma Y = 0$, $\Sigma M = 0$.

$$\Sigma Y = Y_{AK} + Y_{KL} + Y_{LG} + Y_{GF}$$

$$+ Y_{FE} + Y_{EC} + Y_{CB} + Y_{BA} = 0.$$

$$350 - 100 - 100 + 0.$$

$$+ FE \sin 60^\circ + 0 - 100 - 100 = 0.$$

$$350 - 400 + FE \frac{\sqrt{3}}{2} = 0.$$

$$FE = \frac{100}{\sqrt{3}}$$

From the positive sign of the result, the Y_{FE} must be positive.

The member FE is, therefore, in tension $\frac{100}{\sqrt{3}}$ pounds.

Take moments about the intersection of CE and EF at the point O.

$$\Sigma M = M_{AK} + M_{KL} + M_{LG} + M_{GI}$$
$$+ M_{FE} + M_{EC} + M_{CB} + M_{BA} = 0.$$
$$350.25 - 100.15 - 100.5 + GF.5\sqrt{3}$$
$$+ 0 + 0 - 100.10 - 100.20 = 0.$$
$$350.25 - 100\,(15 + 5 + 10 + 20)$$
$$+ GF.5\sqrt{3} = 0.$$
$$GF = -\frac{750}{\sqrt{3}}$$

From the negative sign of the result the moment of GF about O must be negative. GF, therefore, acts away from the section;

$$\text{i.e., the member GF is in tension } \frac{750}{\sqrt{3}} \text{ pounds.}$$

$$\Sigma X = X_{AK} + X_{KL} + X_{LG} + X_{GI}$$
$$+ X_{FE} + X_{EC} + X_{CB} + X_{BA} = 0.$$
$$0 + 0 + 0 + GF + FE.\cos 60^\circ$$
$$+ CE + 0 + 0 = 0.$$
$$\frac{750}{\sqrt{3}} + \frac{100}{\sqrt{3}}.\frac{1}{2} + CE = 0.$$
$$CE = -\frac{800}{\sqrt{3}}.$$

From the negative sign of this result, the X_{CE} must be negative. CE, therefore, acts against the section;

$$\text{i.e., the member CE is in compression } \frac{800}{\sqrt{3}} \text{ pounds.}$$

The student is advised to choose other sections in the same truss and work out the stresses in the various members by this means.

CHAPTER VIII.

THE BEAM.

In the following discussion, the case of a beam supported freely on two abutments, one at each extremity of the beam, will be considered. This type of beam is the one most commonly met with in engineering practice, although cases do arise in which a beam structure is supported upon more than one abutment, or in which the abutment supports are not free, as in the case of a beam having its extremities built into the supporting walls. It is sufficient to point out that these last cases cannot be investigated by the theory in these pages alone, but necessitate a knowledge of the laws of the resistance of materials and of the calculus.

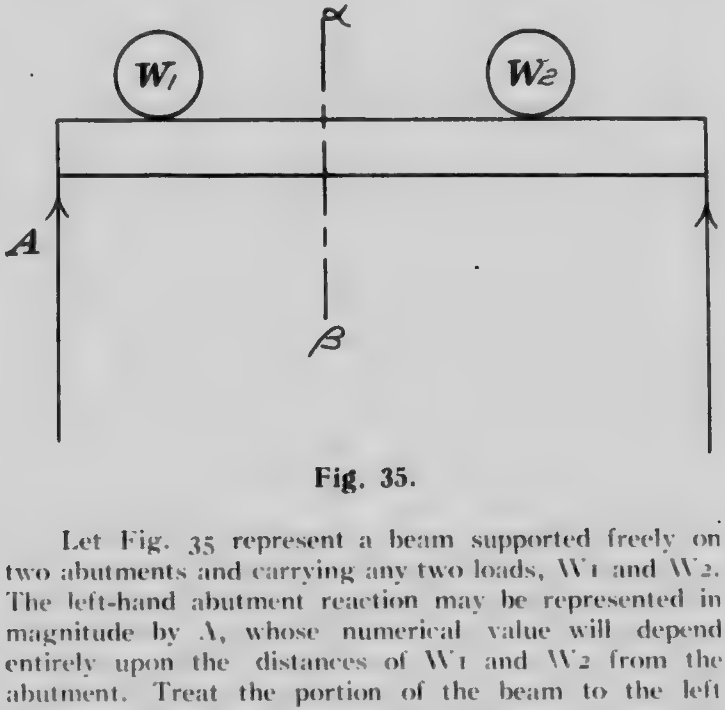


Fig. 35.

Let Fig. 35 represent a beam supported freely on two abutments and carrying any two loads, W_1 and W_2. The left-hand abutment reaction may be represented in magnitude by A, whose numerical value will depend entirely upon the distances of W_1 and W_2 from the abutment. Treat the portion of the beam to the left of the plane $\alpha\beta$ by the Method of Sections. The only **known** forces acting on this portion to the left of $\alpha\beta$ are the reaction A and the load W_1. These forces cannot by themselves maintain equilibrium. It is evident, therefore, that some other unknown force or forces must act upon the portion of the beam being considered. The only possible place where such a force or forces could act is at the section $\alpha\beta$ in the beam. Let Fig. 36 represent the condition of affairs as far as the forces acting upon the body to the left of $\alpha\beta$ are concerned. These forces are in equilibrium; therefore, ΣX, ΣY, and ΣM must each equal zero.

$$\Sigma Y = Y_A + Y_{W1} + (\text{Y's of unknown forces}) = 0.$$
$$A - W_1 + (\text{Y's of unknown forces}) = 0.$$

Replacing the Y's of the unknown forces by their resultant, which may be denoted by V

$$A - W_1 + V = 0.$$
$$V = -(A - W_1).$$

From which it is seen that V is equal in magnitude but acts in the opposite sense to the algebraic sum of the vertical resolved parts of the exterior forces to the left of $\alpha\beta$; i.e., V resists $A - W_1$. This quantity $(A - W_1)$ is called the **Vertical Shearing Force** at the section under consideration.

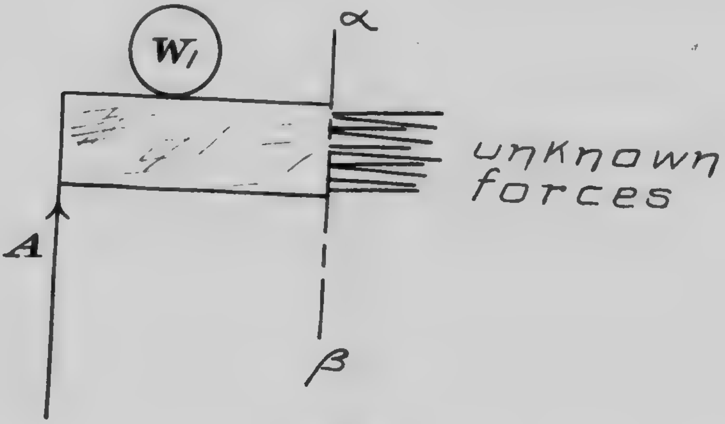


Fig. 36.

Definition:—

The Vertical Shearing Force at any section is equal to the algebraic sum of the vertical resolved parts of the exterior forces acting on the body to the left of the section being considered.

This is a general definition, but since the loads and abutment reactions in most cases, and certainly in all cases that will be taken up here, act in the vertical direction, a more concise definition may be employed.

The Vertical Shearing Force for Vertical Loading and Abutment Reactions is equal to the algebraic sum of all exterior forces to the left of the section being considered.

Applying ΣX to the forces acting as indicated in Fig. 36

$$
\begin{aligned}
\Sigma X = X_A + X_{W_1} + (X\text{'s of unknown forces}) &\quad 0. \\
0 + 0 + (X\text{'s of unknown forces}) &\quad 0. \\
(X\text{'s of unknown forces}) &\quad 0.
\end{aligned}
$$

Now, there are two ways in which the algebraic sum of the X's of the unknown forces may be equal to zero. Either all the X's are equal to zero, or the sum of the positive X's is equal in magnitude to the sum of the negative X's.

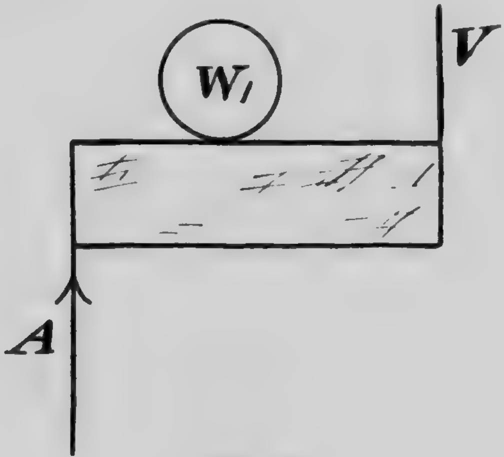


Fig. 37.

In order to determine which of these two cases is the true one, consider the forces acting on the portion of the body to the left of $\alpha\beta$ as they have been deduced so far. Let $W_1 < A$; then V must act downward, for V has been shown to act contrary to $(A - W_1)$, which in this case will be a positive quantity; i.e., V acts with negative sense.

Fig. 37 represents the forces A, W_1, and V acting on the body. It is seen that these forces by themselves cannot give equilibrium; therefore, some other forces must act on the body. It is evident, then, that the X's of the unknown forces are not all equal to zero, for if they were, the only other force which could exist would be a vertical force. But all the vertical forces possible are represented in the diagram. The positive X's must, then, equal the negative X's. Replace the positive and negative X's by their resultants, T and C, respectively. The true condition of affairs must be as represented in Fig. 38.

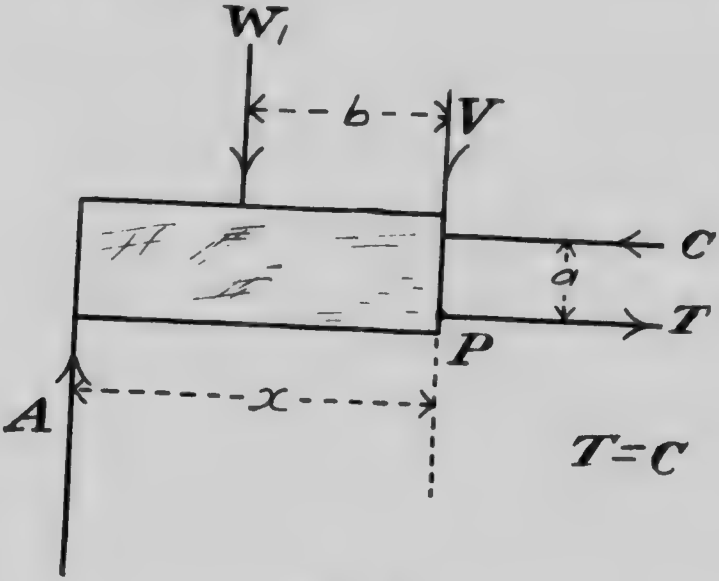


Fig. 38.

C and T form a couple of magnitude C.a. Apply ΣM to the forces acting on the body. Take moments about the point P.

$$\Sigma M = M_A + M_{W_1} + M_V + M_C + M_T = 0.$$

$$\text{But } M_V \text{ and } M_T = 0.$$

$$\text{Therefore, } M_C = -(M_A + M_{W_1}).$$

But $M_C = C.a$ = moment of couple formed by T and C; that is, the moment of this couple is equal in magnitude but opposite in sense to the algebraic sum of the moments of the exterior vertical forces to the left of $\alpha\beta$ about a point in the section.

In this particular case, which is perfectly general, it is seen that M_C resists $(M_A + M_W)$. This quantity $(M_A + M_W)$ is called the **Bending Moment** at the section considered.

Definition:—

The Bending Moment at any section for vertical loading and abutment reactions is equal in magnitude to the algebraic sum of the moments of the exterior forces to the left of the section, about a point in the section.

CHAPTER IX.

VERTICAL SHEARING FORCE AND BENDING MOMENT FOR STATIONARY LOADS ON A BEAM.

The student is particularly advised to memorize the definitions of the Vertical Shearing Force (V.S.F.) and the Bending Moment (B.M.) at any section in a beam as given in the last chapter.

In explanation of the meaning of these terms, V.S.F. and B.M., it may be pointed out that in a beam under a load there is a tendency to bend; and because the deflection due to bending is different at various sections, it is reasonable to suppose that the bending influence is variable for different sections of the beam, which reasoning is borne out by the theory. The B.M. is related to this bending influence. Secondly, any portion of a beam has a tendency to slip past, or **shear** past, the other portion of the beam. For instance, in Fig. 39 the portion A has sheared past B. Whether any given portion of a beam actually does shear or not is aside from the question that there exists such a tendency. The V.S.F. is related to this tendency, which is variable for different sections.

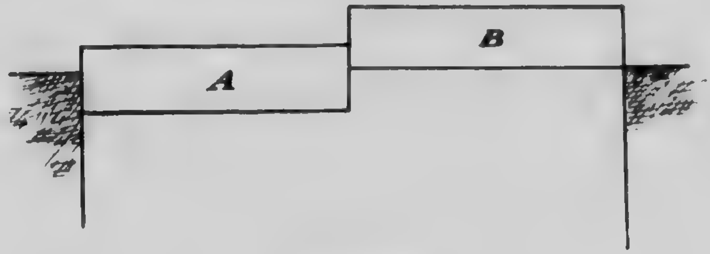


Fig. 39.

To Trace the Variation in V.S.F. Due to a Single Concentrated Load.

In Fig. 40, consider **any** section between the load W and the left-hand abutment. The V.S.F. at any such section = algebraic sum of exterior forces to left of section.

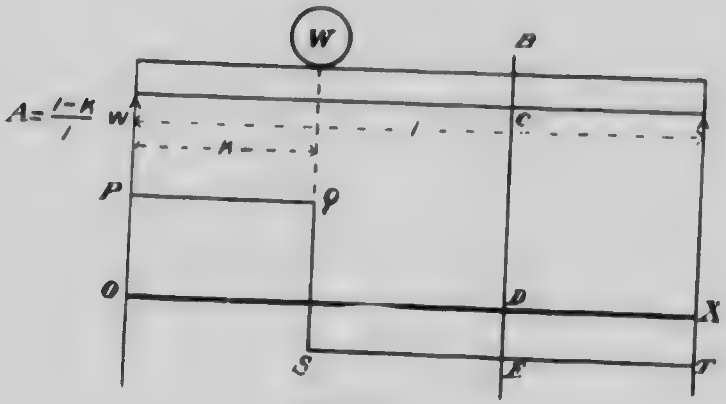


Fig. 40.

The only force to the left of such a section is the abutment reaction $A = \frac{l-K}{l}.W.$

Therefore, V.S.F. for all sections between W and A, $= A$

a positive constant value.

Consider **any** section to the right of W.

V.S.F. at any such section = algebraic sum of exterior forces to left of section.

The algebraic sum of the forces to left of any such section is $(A - W)$.

Therefore, V.S.F. for all sections to right of W = $(A - W)$, and since $W > A$—

V.S.F. = a negative constant value.

Let OX be drawn parallel to the longitudinal axis of the beam.

Let y = V.S.F. at any section of the beam.

For all sections to the left of W, y = positive constant = A.

For all sections to the right of W, y = negative constant = $(A - W)$.

Plotting these ordinates with OX as a base line, and plotting them so that the value of the shear at any section lies directly under that section, the curves will be of the form PQ and ST, two straight lines parallel to OX. The shear at a section immediately underneath W is indeterminate, and may be represented as such by joining Q and S. The diagram PQST is known as the **V.S.F. Diagram.** If it is desired to find the V.S.F. at any section such as BC, all that is necessary is to produce BC to intersect OX and the locus ST at D and E respectively. The ordinate DE represents the V.S.F. at section BC.

To Trace the Variation in V.S.F. Due to any Number of Concentrated Loads.

In this case an actual numerical example will serve the purpose as well as, if not better than, mere general values.

Consider the case of a beam 50 feet long supporting four concentrated loads of magnitude, 100, 200, 50, and 100 pounds, situated at distances 10, 20, 25, and 30 feet, respectively, from the left-hand abutment. Required to draw the V.S.F. diagram.

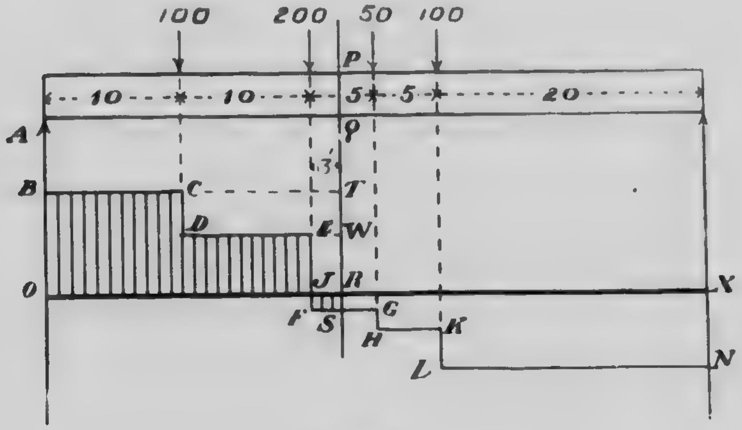


Fig. 41.

In Fig. 41 the left-hand abutment reaction

$$A = 4/5 \cdot 100 + 3/5 \cdot 200 + 1/2 \cdot 50 + 2/5 \cdot 100.$$
$$= 265 \text{ pounds.}$$

Let y = V.S.F. at any section.

Between A and first load, $y = A = 265$.

Between first and second load, $y = 265 - 100 = 165$.

Between second and third load,

$$y = 265 - 100 - 200 = -35.$$

Between third and fourth load,

$$y = 265 - 100 - 200 - 50 = -85.$$

Between fourth load and right-hand abutment,

$$y = 265 - 100 - 200 - 50 - 100 = -185.$$

Plotting these ordinates with OX as base line, a V.S.F. diagram of the form BC, DE, FG, HK, LN is obtained, such that the ordinate to the diagram immediately beneath any section of the beam represents the value of the V.S.F. at that section.

It will be noticed that **the change in value of shearing force** from a section immediately to the left of any load to a section just to the right of the load **is equal to the magnitude of the load itself.** For instance, in the last numerical case, the changes represented by CD, EF, GH, and KL are each equal in magnitude to the loads immediately above these lines.

Representation of Bending Moment at any Section of a Beam.

Let PQ (Fig. 41) be the section, the B.M. at which it is required to represent. PQ produced intersects the V.S.F. diagram in RS.

Let PQ be three feet to the right of the 200-pound load.

Produce BC and DE to intersect PQ at T and W.

B.M. at PQ = A × 23 — 100 × 13 — 200 × 3.
But A × 23 = OB × BT = area OT.
100 × 13 = CD × CT = area DT.
200 × 3 = EF × EW = area FW.
Therefore, B.M. = area OT — area DT — area FW.
= area OBCDEJ — area JFSR.
area of V.S.F. diagram to left of PQ.

It must, of course, be remembered that areas above OX are considered positive and those below OX negative. Keeping this last statement in mind, it is seen that—

The area of the V.S.F. diagram to the left of any section represents the B.M. at that section.

The B.M. at any section is always positive, for the positive area of the V.S.F. diagram to the left of any section is, with one exception, always greater than the negative area to the left of the same section. It may be proven geometrically that the positive and negative areas to the left of the right-hand abutment are equal; hence, the B.M. at this abutment is zero. The B.M. at the left-hand abutment is also zero, since there is no area of the V.S.F. diagram to the left of that section.

Section at which Maximum B.M. Occurs for a Single Load.

ABDF (Fig. 42) is the V.S.F. diagram for the load W. Consider areas above OE positive, below OE negative.

The B.M. at any section NP to the left of the load is represented by the area ANPO.

The B.M. at any section GL, to the right of the load, is represented by the area ABCO — CGLD.

Both of these areas are less than the positive area ABCO. Therefore, the area ABCO must represent the greatest B.M.

But ABCO represents the B.M. at a section infinitely close to BC, but still to the left of the load. Practically, this is the section BC. It follows that:—

The Maximum B.M. for a single load in a given position occurs at a section directly under the load.

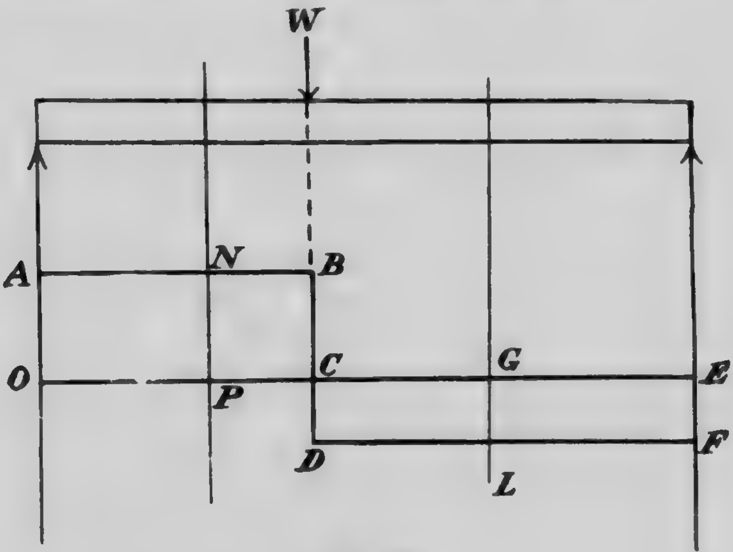


Fig. 42.

Uniformly Distributed Load.

A uniformly distributed load is one such that, no matter how small a portion of the area loaded is considered, the load per **unit area** is always the same.

Since the discussion is confined to co-planar forces, a uniformly distributed load in the following problems will mean a load of so much per **unit length**.

It may be pointed out that in practice the term "uniformly distributed load" is very often applied to a number of concentrated loads placed very close to one another. For instance, a crowd of people standing on a floor is referred to as a uniformly distributed load of so much per square foot, when in reality the load is made up of a number of concentrated loads placed very close to one another. However, the assumption gives results well within all limits of safety.

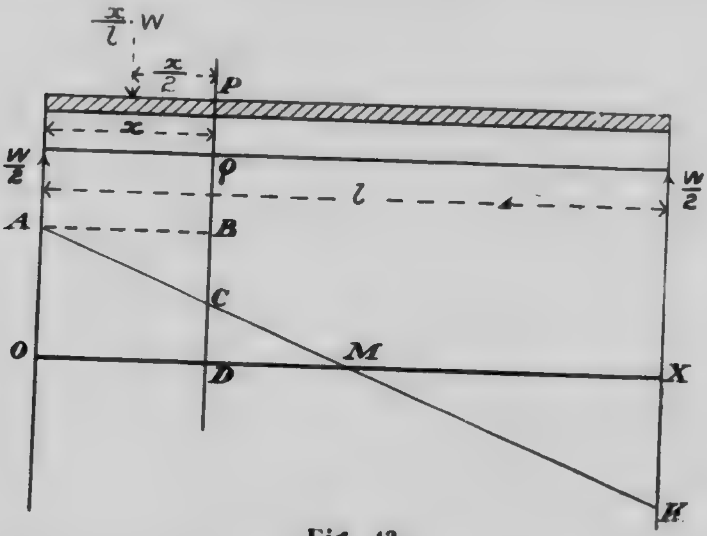


Fig. 43.

To Trace the Variation of V.S.F. and B.M. in a Beam Supporting a Uniformly Distributed Load of W Pounds.

Let the hatched area of Fig. 43 represent the load uniformly distributed over the entire length of the beam.

Total load = W lbs. Abutment reactions $= \frac{W}{2}$.

$$\text{Load per unit length} = \frac{W}{l}.$$

Take **any** section PQ at a distance x from the left-hand abutment.

V.S.F. at PQ = y = (left-hand abutment reaction) — (load to left of PQ).

$$\text{Load to left of PQ} = \frac{W}{l}. x.$$

$$\text{Therefore, } y = \frac{W}{2} - \frac{W}{l}.x.$$

If the values of y be plotted with OX as axis of x, the curve will be a straight line such that, as in the previous cases, the value of the V.S.F. at any section will be directly under that section. This is shown by the line AK.

$$\text{To trace the curve, } y = \frac{W}{2} - \frac{W}{l}.x \ldots\ldots\ldots\ldots (1.)$$

$$\text{When } x = 0,\ y = \frac{W}{2} = OA.$$

$$\text{When } x = l,\ y = -\frac{W}{2} = XK.$$

$$\text{When } y = 0,\ x = \frac{l}{2} = OM.$$

From this it is seen that the V.S.F. varies from a positive value of $\frac{W}{2}$ at the left-hand abutment, to a value $-\frac{W}{2}$ at the right-hand abutment, being equal to zero at the centre of the beam.

In order to consider the variation in B.M., let the load to the left of **any** section PQ be replaced by an equivalent concentrated load of magnitude $\frac{x}{l}.W$, which will act a distance $\frac{x}{2}$ from PQ as dotted in Fig. 43.

$$\text{B.M. at PQ} = \frac{W}{2}.x - \frac{W}{l}.x\frac{x}{2} \ldots\ldots\ldots\ldots (2.)$$

Produce PQ to intersect the V.S.F. diagram at C and D. Draw AB perpendicular to PQ.

OD and DC are co-ordinates to a point on AK below PQ.

$$OD = x ; \ y = DC.$$

Therefore, from (1.), $y = DC = \frac{W}{2} - \frac{W}{l}.x$ . . . (3.)

From the figure it is seen that

$CB = DB - DC = OA - DC$ . . . . (4.)

$$\text{But } OA = \frac{W}{2}.$$

Substitute in (4.) this value of OA, and also the value of DC found in (3.).

$$\text{Then, } CB = \frac{W}{2} - \left(\frac{W}{2} - \frac{W}{l}.x\right).$$

$$= \frac{W}{l}.x.$$

Tabulating these results—

$$\frac{W}{2} = OA.$$

$$x = OD = AB.$$

$$\frac{W}{l}.x = CB.$$

Substituting these values into (2.).

$$B.M. = OA \times OD - CB \times \frac{AB}{2}.$$

= Area OB — △ ABC.
= Area OACD.
= Area of V.S.F. diagram to left of section.

The B.M. evidently varies from a value of zero at the left-hand abutment to zero at the right-hand abutment, having always positive values.

It is seen that **the maximum B.M. for a beam supporting a uniformly distributed load over its entire length is at the centre of the beam**, for the area AMO is the greatest positive area.

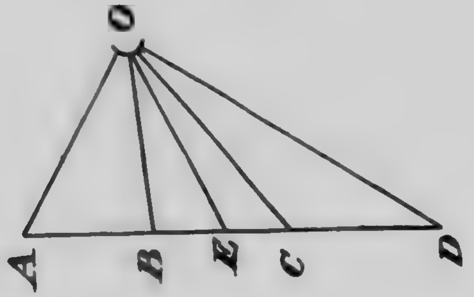


Fig. 44 b.

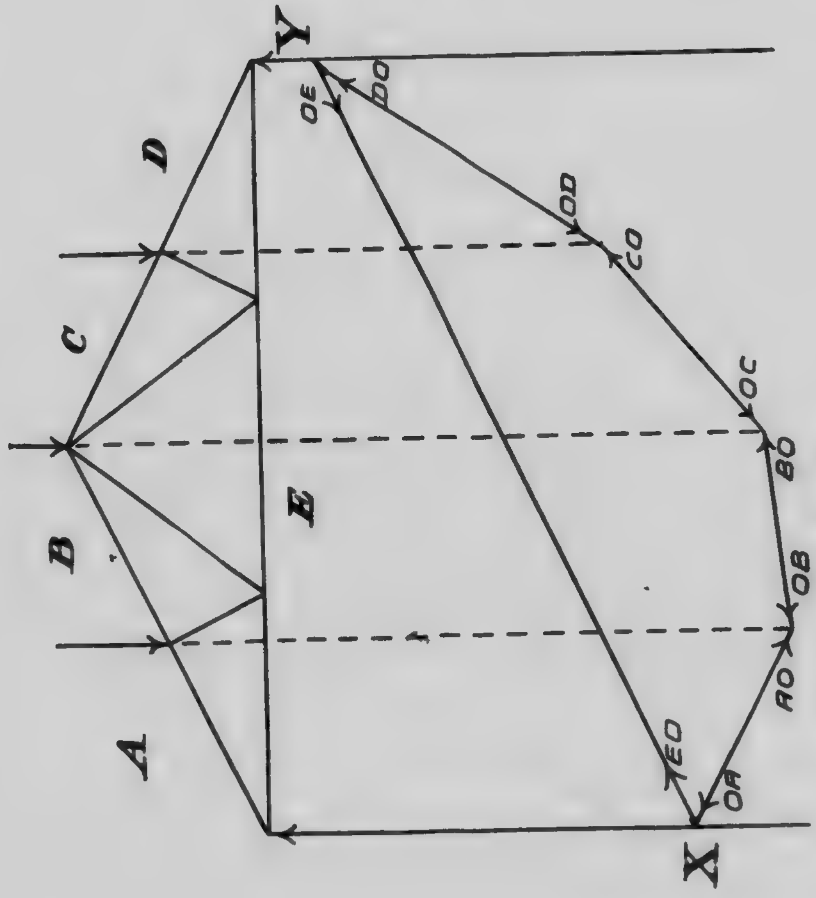


Fig. 44 a.

Graphical Determination of Abutment Reactions.

In the work previous to this, the abutment reactions whenever desired have been calculated analytically. There is, however, a graphical method of determining these reactions, the procedure being as follows:—

Let Fig. 44 a represent any frame structure resting freely on two abutments and supporting the loads AB, BC, CD of any magnitude whatever. These loads, together with the unknown abutment reactions DE and EA, give equilibrium.

The vector polygon, AB, BC, and CD (Fig. 44 b), may be drawn to represent the three loads, and, since the forces acting on the structure are in equilibrium, the vector polygon must close. The only forces that can close it are the two abutment reactions. It follows, then, that between D and A there lies some point E, such that DE and EA represent these two reactions. If E can be located, the problem is solved.

Choose any pole O in Fig. 44 b and join O to A, B, C, and D. As far as possible construct the funicular polygon for the forces acting on the structure, commencing by replacing AB by AO and OB. Proceed by replacing BC with BO and OC, and CD by CO and OD. Produce the lines of action of AO and OD to intersect the abutment reactions at X and Y, respectively. Now, no matter where the point E lies between D and A in Fig. 44 b, the components of DE and EA will be DO, OE, and EO, OA, respectively. If the abutment reactions were replaced at Y and X by these components, it is seen that the only possible manner in which OE and EO, which are as yet unknown, can act in order that the funicular polygon may close, is in the line XY. This determines the point E, for in order that the components OE and EO may act in the direction XY, the line joining O to E must be parallel to XY; i.e., **to locate E, through O draw OE parallel to XY, intersecting DA at E.** DE and EA (Fig. 44 b) represent the required reactions.

Relation between Bending Moment and the Funicular Polygon.

Let Fig. 45 represent a beam supporting any general loading, AB, BC, and CD. To the right of the figure is the vector polygon, AB, BC, CD, DE, and EA, the abutment reactions DE and EA having been found graphically by means of the funicular polygon, as explained in the last problem. It is required to find the value of the B.M. at any section (1.).

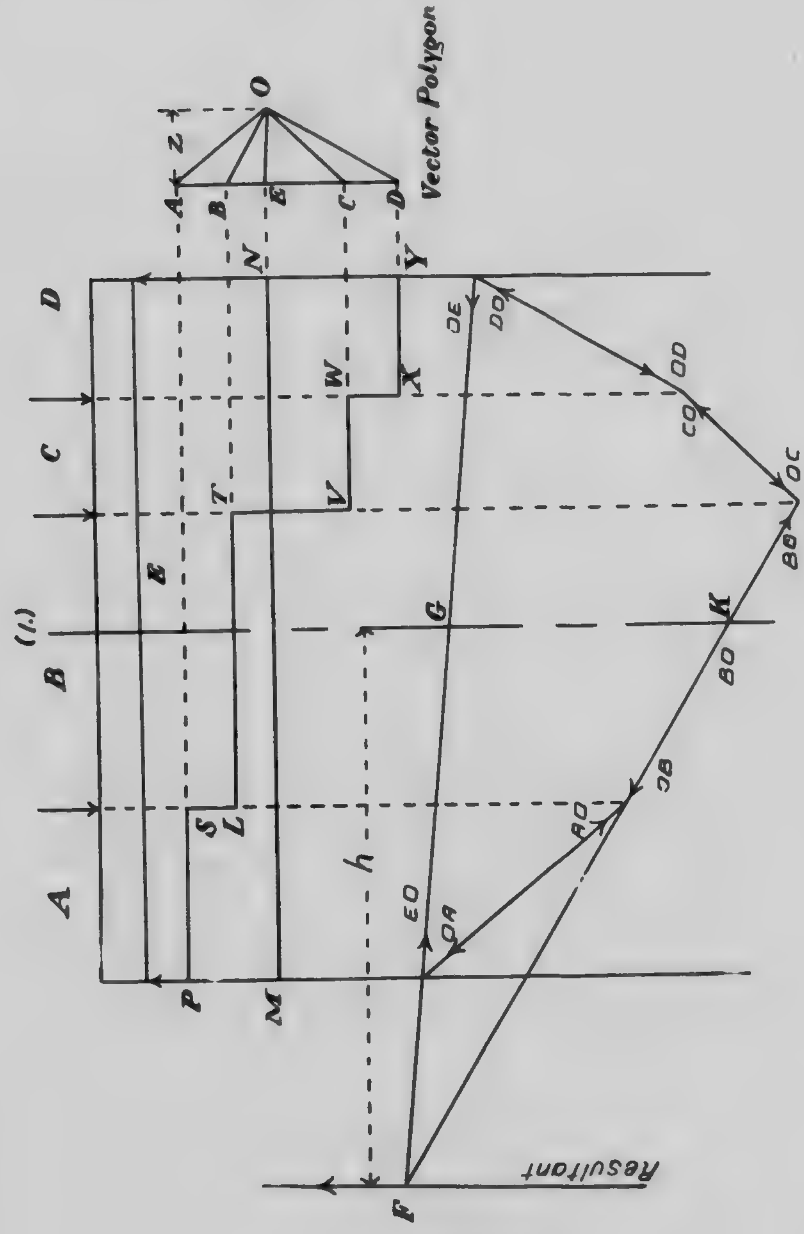


Fig. 45.

Produce (1.) to intersect the funicular polygon at G and K. Let the perpendicular distance of the pole O from AB be Z.

The B.M. at (1.) = Algebraic sum of the moments of EA and AB about any point in (1.).
= Moment of the resultant of EA and AB about any point in (1.).

To locate the resultant of EA and AB; i.e., the resultant of all forces to the left of plane (1.):

EA has been replaced by the components EO and OA as shown in the funicular polygon, and AB by the components AO and OB. It is seen that OA and AO counteract one another, which leaves EO and OB as equivalent to the forces EA and AB; i.e., the resultant of EO and OB is the resultant of EA and AB. But from the vector diagram, EB represents the resultant of EO and OB, which will act at the intersection of the lines of action of EO and OB as indicated at the point F. The moment of this resultant about a point in section (1.) is the required B.M. Let h be the distance of the line of action of the resultant from section (1.).

Now,
$$FG \parallel EO.$$
$$FK \parallel OB.$$
$$GK \parallel BE.$$

Therefore, Δ's FGK and BEO are similar.

Then,
$$\frac{EB}{GK} = \frac{\text{altitude } z.}{\text{altitude } h}.$$

Or, $EB \times h = GK \times z$ .... (3.)

But EB represents the resultant of the forces to the left of section (1.).

Therefore, $EB \times h = B.M.$ .... (4.)

i.e., from (3.) $B.M. = GK \times z$................. (5.)

Now, GK is called the ordinate of the funicular polygon at section (1.), and it is seen that the B.M. is represented by this ordinate multiplied by a constant quantity z; so that, **provided the proper scale is used, the B.M. at any section is represented by the ordinate of the funicular polygon immediately beneath the section.**

To Find the Scale of B.M.'s to be Used with Ordinates of the Funicular Polygon.

From (4.) in the last discussion—

B.M. = EB × h.
= (EB in inches × scale of forces) (h in inches × scale of dimensions).
= EB × h (scale of forces × scale of dimensions).

But from (3.) EB × h = GK × z.

Therefore, B.M. = GK (z × scale of forces × scale of dimensions); i.e., the scale to use on GK in order to find the actual value of the B.M. is **(z × scale of forces × scale of dimensions)**, z being taken in inches if the scales are per inch.

Caution:

In order to get a good working scale of B.M.'s, the pole O should be so chosen with reference to AD, that z will be a quantity which will, when multiplied by the scales of forces and dimensions, give an even quantity suitable for use with ordinary scales.

A Graphical Method of Constructing the V.S.F. Diagram for Concentrated Loads.

Let the vector diagram of the forces acting on the beam be drawn to one side and slightly below the representation of the beam as shown in Fig. 45. It will be assumed that the abutment reactions DE and EA have been found. From E draw a line ENM parallel to the longitudinal axis of the beam, and intersecting the abutment reactions at N and M.

The V.S.F. at any section between the left-hand abutment and the first load is equal to the left-hand abutment reaction, the value of which reaction is represented by EA. Therefore, if the scale of shearing forces be chosen the same as the scale of forces used in constructing the vector diagram, the ordinates from MN to the line PS, which is drawn parallel to MN through A, will represent the V.S.F. values at any section between the left-hand abutment and the first load, for these ordinates will each be equal to EA, which is the value of the V.S.F. between the sections indicated.

The V.S.F. for any section between the first and second load is equal to the left-hand abutment reaction minus the first load AB. Graphically, this is represented by

$$EA - AB = EB.$$

Therefore, LT, drawn parallel to MN through B, gives a diagram, the ordinates from MN to which, represent the V.S.F. values between the first and second load.

Following the same line of reasoning, the ordinates from MN to VW and XY, which are drawn parallel to MN from C and D, respectively, represent the V.S.F. values from the second load to the right-hand abutment.

CHAPTER X.

V.S.F. AND B.M. FOR MOVING LOADS ON A BEAM.

To Trace the Variation in V.S.F. and B.M. for a Single Moving Load.

Let a load of W pounds travel over the beam to a position distant x from the left-hand abutment as represented in Fig. 46. The left-hand abutment reaction A will then be $\frac{l-x}{l}.W$.

RBDH is the V.S.F. diagram for the load in this position.

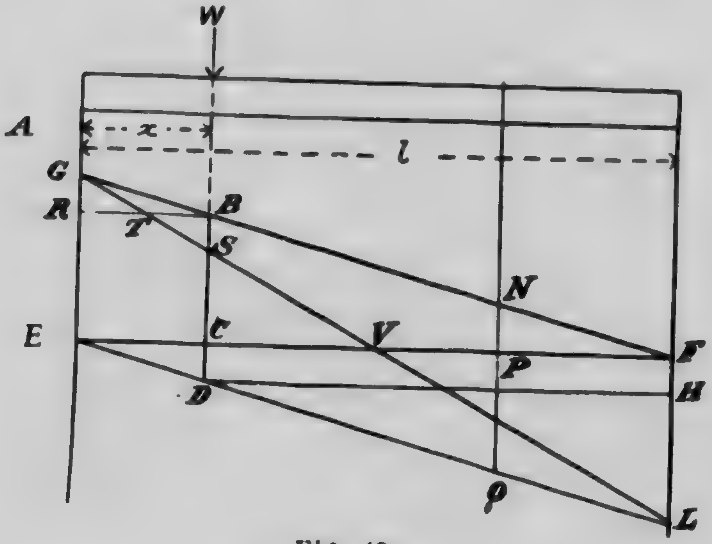


Fig. 46.

Now, since the load is a moving one, it would be extremely inconvenient to draw a series of V.S.F. diagrams for many or all positions of the load.

Looking at the diagram RBDH, which is a perfectly general one, it is seen that because RB and DH are parallel to EF, BC represents the value of the V.S.F. at any plane to the left of the load and CD the V.S.F. at any plane to the right of the load.

But V.S.F. at any plane to left of load = A $\frac{l-x}{l}.W = BC$; and V.S.F. at any plane to right of load $= A - W = \frac{l-x}{l}.W - W = CD$.

If, then, ordinates $y = \frac{l-x}{l}.W$ be plotted underneath the load as it moves to different positions, such that y represents the value of the V.S.F. at any plane to the left of the load, and ordinates $y' = \frac{l-x}{l}.W - W$ be also plotted underneath the load in the same series of positions, so that y′ represents the V.S.F. at all planes to the right of the load, the locii of the points so obtained will be the straight lines GF and EL, respectively. (y′ will, of course, be negative for $W > \frac{l-x}{l}.W$.)

GFLE is a V.S.F. diagram for the moving load W. Supposing it is desired to find the V.S.F. at a plane NQ: (1.) when the load is over BD; (2.) when the load is over NQ.

(1.) The V.S.F. at NQ when the load is over BD is represented by CD, for CD represents the V.S.F. for **all** planes to right of load when load is over BD.

(2.) When the load is directly over NQ, the V.S.F. at NQ is indeterminate, but NP represents the V.S.F. at all planes to left of NQ, and PQ represents the V.S.F. at all planes to right of NQ. These results may be arrived at analytically by letting x be the distance of the load from the left-hand abutment.

For any plane to the left of the load—

$$V.S.F. = y = \frac{l-x}{l}.W.$$

For any plane to the right of the load—

$$V.S.F. = y' = \frac{l-x}{l}.W - W.$$

To Find the Maximum Possible B.M. as the Load Moves Over Beam.

For the load in a given position it has been shown that the maximum B.M. is at a section directly under the load. It is only necessary to compare these maximum B.M.'s for the load in different positions in order to find the greatest possible B.M.

Taking the load in a position represented in Fig. 46, the maximum B.M. for this position is represented by RBCE.

Join GL.

The area GSCE is then equal to the area RBCE, for the area RTSCE is common to both, and the Δ GRT equals the Δ TBS.

Therefore, the area GSCE represents the maximum B.M. for the load in that position. In the same way, for the load in other positions, the maximum B.M. is represented by the area between GL and EF to the left of the section under the load.

It is seen that the greatest B.M. that will occur is represented by the positive area GVE. This area represents the B.M. when the load is at the centre of the beam.

Hence:

The maximum possible B.M. for a single moving load occurs directly under the load when it is at the centre of the beam.

CHAPTER XI.

PULLEY SYSTEMS.

It is the intention, in the following discussion, to find out how one or more pulley blocks may be most advantageously arranged in order to lift or move a given weight.

In most of the problems the ropes of the system are assumed to be parallel to one another. This, of course, is not quite true in practice, but the error arising from such an assumption is so slight that it may be neglected, together with the errors that arise from assuming that the sheaves have frictionless bearings and that their weights are negligible.

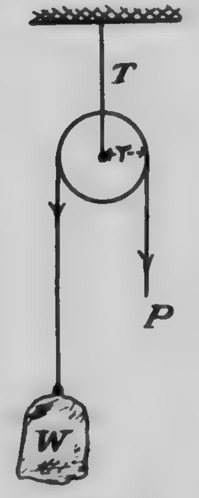


Fig. 47.

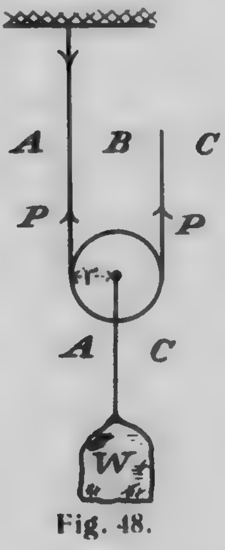


Fig. 48.

Mechanical Advantage.

The mechanical advantage of a system of pulleys is the ratio of the weight lifted to the force exerted on the free end of the rope.

The mechanical advantage is sometimes referred to as the **efficiency of the system.**

I. Let Fig. 47 represent a single block fastened to some support. At one end of the rope is fastened a weight, W. It is required to find the force P which must be exerted at the free end of the rope in order to keep the weight W in equilibrium. Any greater force than P will lift W.

Consider the pulley as a rigid body. It is acted upon by three forces, P, W, and T, in equilibrium.

Take moments about the centre of rotation of the sheave.

$$\Sigma M = M_P + M_W + M_T = 0.$$
$$P.r - W.r + 0 = 0.$$
$$P = W;$$

i.e., with such an arrangement of the block the force exerted at the free end of the rope to keep the system in equilibrium must be equal to the weight.

$$\text{Mechanical Advantage} = \frac{W}{P} = 1.$$

II. Let the block be attached to the weight and one end of the rope made fast to some abutment as represented in Fig. 48.

Consider the pulley as a rigid body. It is acted upon by the three forces AB, BC, and CA (using Bow's Notation) in equilibrium.

Take moments about the axis of rotation as in I.

$$\Sigma M = M_{AB} + M_{BC} + M_{CA} = 0.$$
$$AB.r - BC.r + 0 = 0.$$
$$AB = BC.$$

If a force of P pounds be applied at the free end of the rope, then BC = P = AB.

To find the magnitude of P:

$$\Sigma Y = Y_{AB} + Y_{BC} + Y_{CA} = 0.$$
$$P + P - W = 0.$$
$$P = \tfrac{1}{2} W.$$

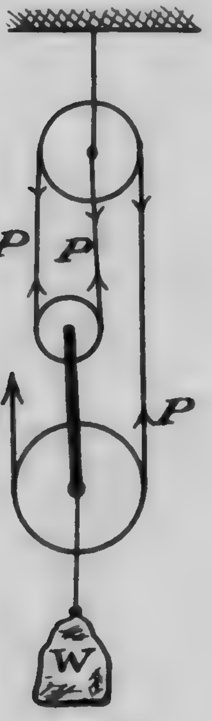
$$\text{Mechanical Advantage} = \frac{W}{P} = \frac{W}{\frac{1}{2}W} = 2.$$

III. In a system such as represented by Fig. 49, if a force P be applied to the free end of the rope, there will exist a tension of P pounds throughout the rope system.

The two lower blocks are connected together rigidly. They may, then, be considered as a body acted upon by four upward forces of magnitude P, due to the tension in the ropes, and a fifth downward force of magnitude W. These forces are in equilibrium.

$$\Sigma Y \text{ for lower two blocks} = 4P - W = 0.$$
$$P = \tfrac{1}{4} W.$$

$$\text{Mechanical Advantage} = \frac{W}{P} = 4.$$



No. 49.

The following cases, it is easily seen, are merely special examples of the preceding problems. For this reason the solution of each case is condensed.

The inverse arrangement of the last case does not give as good efficiency. This system is shown at Fig. 50.

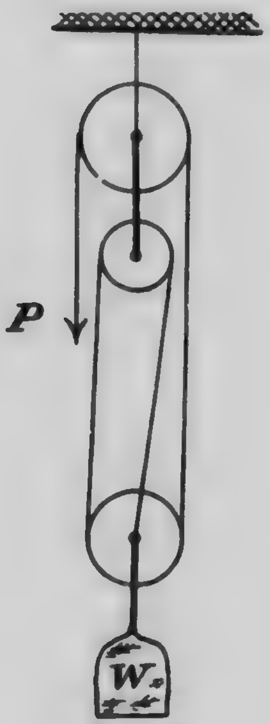


Fig. 50.

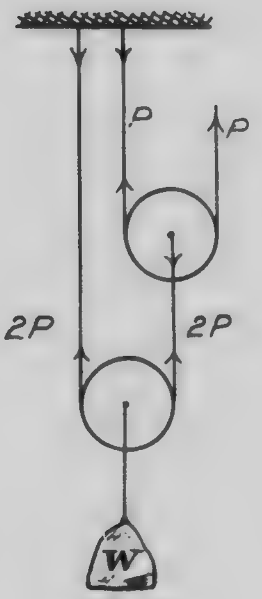


Fig. 51.

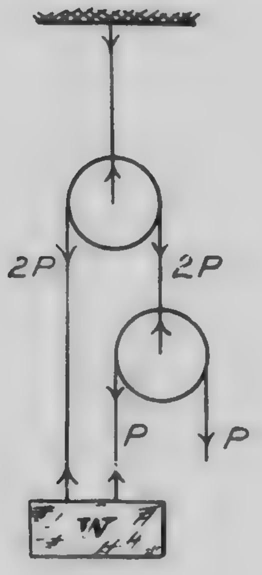


Fig. 52.

$$\Sigma Y \text{ for lower block} = 3P - W = 0.$$
$$P = \tfrac{1}{3}W.$$
$$\text{Mechanical Advantage} = \frac{W}{P} = 3.$$

Referring to Fig. 51, if P be the magnitude of the force applied to the free end of the rope, then the tension in the rope passing over the lower pulley will be 2 P as indicated.

$$\Sigma Y \text{ for lower block} = 2P + 2P - W = 0.$$
$$P = \tfrac{1}{4}W.$$
$$\text{Mechanical Advantage} = \frac{W}{P} = 4.$$

In an arrangement such as represented by Fig. 52, if P pounds be applied to the free end of the rope, the mechanical advantage will be 3. The student is advised to figure this out for himself.

The Weston Differential Pulley.

In the ordinary pulley systems, where there is a free rope end, there is always more or less inconvenience arising from the large amount of rope lying around. Added to this is the fact that in order to lift a weight of any size, a number of blocks must be used. To obviate all this, the Differential Pulley is used. Fig. 53 illustrates diagrammatically the simplest form of this type of pulley, upon which is based all other differential arrangements of blocks. This type is known as the Weston Differential Pulley. The system consists of an upper block composed of two sheaves of different radii, cast integral, and a lower block, whose radius is generally about that of the smaller of the upper two, although this is immaterial. The upper two blocks have roughened rims to prevent the endless chain, which passes over the system, from slipping. There is always some arrangement cast on the upper block casing which allows the chain to run when pulled either way, but if left hanging idle, the chain checks. Thus it is possible to lift or lower a weight to any desired height and leave it unattended. The lower block has a smooth circumference.

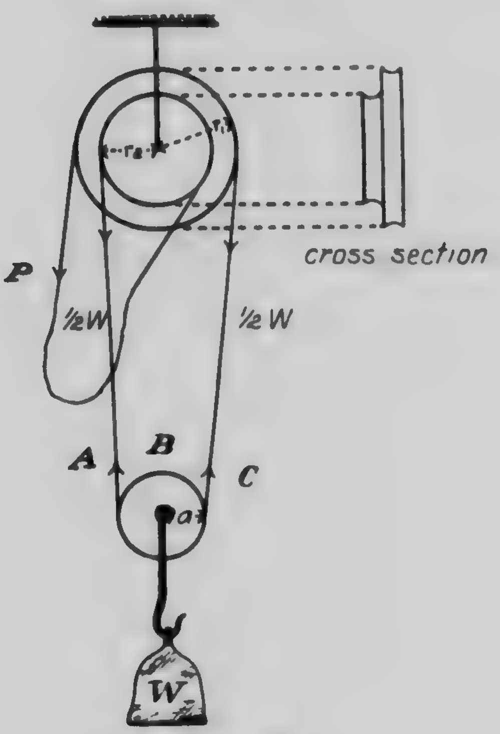


Fig. 53.

For lower block, take moments about axis of rotation.

$$\Sigma M = M_{AB} + M_{BC} + M_{W} = 0.$$
$$AB.a - BC.a + 0 = 0.$$
$$AB = BC \ldots\ldots\ldots\ldots (1.)$$
$$\Sigma Y = Y_{AB} + Y_{BC} + Y_{W} = 0.$$
$$AB + BC - W = 0 \ldots\ldots\ldots\ldots (2.)$$

Substitute for AB from (1.) into (2.)

$$2\ BC = W.$$
$$BC = AB = \frac{W}{2}.$$

Let a force of P pounds be applied to free end of chain.

For the upper block, taking moments about axis of rotation :—

$$\Sigma M = P.r_1 + \frac{W}{2}.r_2 - \frac{W}{2}.r_1 = 0.$$

$$P = \frac{W}{2}\frac{(r_1 - r_2)}{r_1}.$$

$$\text{Mechanical Advantage} = \frac{W}{P} = \frac{2r_1}{r_1 - r_2}.$$

The Mechanical Advantage of such a system is then equal to twice the radius of the large block divided by the difference between the radii of the large and small block.

To Design a Differential Pulley.

Design a differential pulley to lift a ton weight by applying ten pounds force to the free end of the system.

$$\text{Mechanical Advantage} = \frac{W}{P} = \frac{2000}{10} = \frac{2r_1}{r_1 - r_2}.$$

$$\text{Simplifying}: \frac{2000}{10} = \frac{2r_1}{r_1 - r_2}.$$

$$\frac{r_1}{r_2} = \frac{200}{198},$$

so that any differential pulley whose upper blocks have their radii in the ratio of 200:198 will satisfy the demands of the problem.

What would the Mechanical Advantage be if the radii of the two upper blocks of a differential pulley were equal?

Answer: Infinite. The key to this seeming paradox is that, although the advantage is infinite, it would take an infinite time to accomplish anything with such a pulley.

Important.

In the preceding discussion of pulley systems, it is seen that it is possible to lift a large weight by the application of very little force. The student must not fall into the trap of thinking that by doing comparatively little **work** at one end of the system that a large amount of **work** can be done at the other end. The work done at both ends of the system is exactly the same in theory. Practically, there will be a mechanical loss due to friction, etc., between the free end and the weight being lifted.

CHAPTER XII.

FRICTION.

The term **Friction** is used to designate the resistance offered when we attempt to move, or actually do move, one body which rests in contact with another. It is a variable quantity, depending on circumstances, but usually reaches its maximum value when the body is just on the point of motion. The name **Limiting Friction** is used to denote this maximum value.

There are two kinds of friction, **Sliding** and **Rolling**, the former name being applied to cases where a body slides over another, and there is a definite plane of contact; the latter to cases where wheels, etc., roll along with small areas of contact.

One general law which applies to all kinds of friction states that—

Friction always acts in such a way as to oppose motion.

The following cases should be examined and the directions in which friction acts fully determined:

1. Ladder resting against a wall.
2. Iron wheel rolling on an elastic plane.
3. Rubber-tired wheel rolling on an iron plane.
4. Rope with a weight on each end, one greater than the other, stretched over a fixed pulley. (a.) Rope just on the point of moving. (b.) Rope moving.
5. Rod resting on a table pulled at one end perpendicularly to its length.
6. Ring laid on a rough, horizontal table and pulled at any point tangential to ring so that ring is just on point of motion.

The laws of rolling friction have not been fully investigated, and will not be discussed here. Those of sliding friction are more fully understood, being based, however, on experiment, and being true only within certain limits.

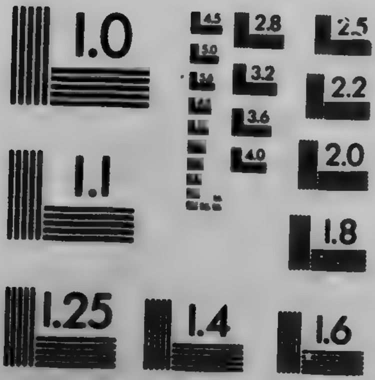
The chief laws are:—

1. When motion is just about to take place, there is an approximately constant ratio between the friction and the **normal pressure** between the two bodies, which is called the Coefficient of Statical Friction.

This ratio is usually a small fraction, and tables are constructed empirically, giving values for different bodies.

MICROCOPY RESOLUTION TEST CHART

(ANSI and ISO TEST CHART No. 2)



APPLIED IMAGE Inc
1653 East Main Street
Rochester, New York 14609 USA
(716) 482 - 0300 - Phone
(716) 288 - 5989 - Fax

2. When motion takes place, the coefficient is smaller than in the former case, and is termed the Coefficient of Dynamical Friction. This statement is true only for small velocities.

3. Friction depends upon the materials used and on their temperature.

Let Fig. 54 represent a body in equilibrium resting on an inclined plane, its weight W acting as indicated through the centre of gravity. There is also acting on the body the reaction, N, of the plane perpendicular to the surface of contact and the friction F. These forces acting on the body are in equilibrium. Therefore, $\Sigma X = 0$; $\Sigma Y = 0$; and $\Sigma M = 0$.

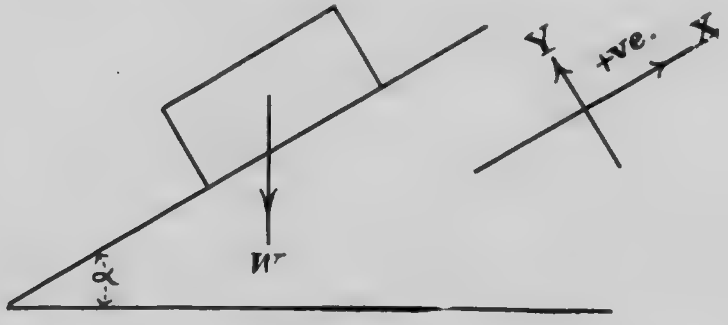


Fig. 54.

Taking the axis of X parallel to surface of plane as indicated—

$$\Sigma X = -W \cos (90 - \alpha) + F = 0.$$

$$F = W \sin \alpha.$$

$$\text{or } W = \frac{F}{\sin \alpha} \quad \ldots\ldots\ldots \quad (1.)$$

$$\Sigma Y = -W \sin (90 - \alpha) + N = 0.$$

$$N = W \cos \alpha \ldots\ldots \quad (2.)$$

Substitute value of W from (1.) into (2.).

$$N = \frac{F}{\sin \alpha} . \cos \alpha.$$

$$\text{or } \frac{F}{N} = \tan \alpha \ldots\ldots\ldots \quad (3.)$$

Now, imagine the plane as being gradually tipped up so that the angle of inclination α becomes greater and greater. A position will be eventually reached beyond which the plane cannot be raised without causing the body to slip. The plane is then said to be inclined at the "**angle of repose.**"

Definition:—

If a body rest on a plane surface, the Angle of Repose is that angle beyond which the plane cannot be inclined without the body moving down the plane.

When the plane reaches the angle of repose from (3.)—

$$\frac{F}{N} = \text{tan. angle of repose.}$$

$\frac{F}{N}$, in this particular instance, is known as the **coefficient of friction,** usually designated by the Greek letter μ.

Coefficient of Friction $= \mu =$ tangent of angle of repose.

From the previous discussion it is seen that if a body rest on a plane surface which is inclined at an angle less than the angle of repose (sometimes called the angle of friction), the body will not slip down the plane. If now, some external force be applied to the body so as to cause the body to be **just on the point of moving** either up or down the plane, the resistance offered by friction can in every such case be experimentally shown to be of a magnitude given by

$$F = N.\mu \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4.)$$

Where F = frictional resistance.
N = pressure normal to the plane.
μ = coefficient of friction for plane and body.

Problems.

A body weighing 100 pounds rests on a plane surface which is inclined to the horizontal at an angle of 30°, the angle of repose for the plane and body being 45°. Find what force acting parallel to the plane will cause the body to be on the point of moving: (1.) up the plane; (2.) down the plane.

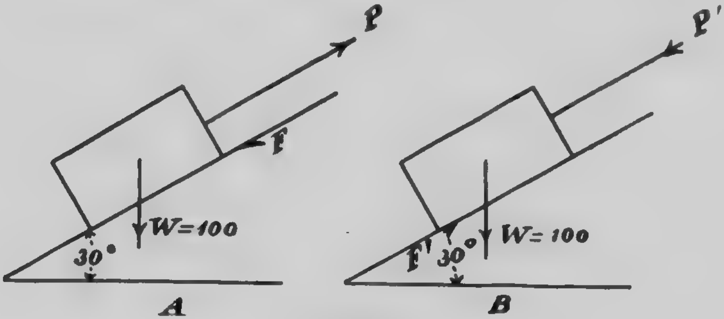


Fig. 55.

Fig. 55 A represents case (1.).

The forces known to act on the body are, W = 100 pounds acting through the centre of gravity of the body; F, the friction; P, the required force.

Since these forces are in equilibrium, $\Sigma X = 0$; $\Sigma Y = 0$; $\Sigma M = 0$.

$$\Sigma X = X_W + X_P + X_F = 0.$$

$$- W \cos 60^\circ + P - F = 0 \ldots\ldots\ldots\ldots\ldots\ldots \quad (5.)$$

Now $F = N.\mu$.

Where $N = W \cos 30^\circ = 100.\frac{\sqrt{3}}{2} = 50\sqrt{3}$.

and μ = tan angle of repose = $\tan 45^\circ = 1$.

Therefore, $F = 50\sqrt{3} \times 1$.

Substituting this value of F into (5.),

$$- 100.\tfrac{1}{2} + P - 50\sqrt{3} = 0.$$

$$P = (50\sqrt{3} + 50) \text{ pounds}.$$

Referring to Fig. 55 B for case (2.), P′ represents the required force, and F′, which in this case acts upward or against the tendency to motion, the friction.

$$\Sigma X = X_W + X_{P'} + X_{F'} = 0.$$

$$- W \cos 60^\circ - P' + F' = 0 \ldots\ldots\ldots\ldots\ldots\ldots \quad (6.)$$

$F' = N.\mu = 50\sqrt{3}$, which if substituted in (6.) gives

$$- 100.\tfrac{1}{2} - P' + 50\sqrt{3} = 0.$$

$$P' = 50\sqrt{3} - 50.$$

A body weighing 50 pounds rests on a plane surface which is inclined to the horizontal at an angle of 45°. The coefficient of friction for body and plane is $1/\sqrt{3}$. What is the angle of repose? Find what force must be exerted on the body in a direction parallel to the plane to keep the body just on the point of sliding up.

The coefficient of friction = tangent of angle of repose

$$\frac{1}{\sqrt{3}} = \tan 30^\circ;$$

i.e., the angle of repose in this case is 30°.

It is evident that the body would, if left to itself, slip down the plane because the plane is inclined past the angle of repose.

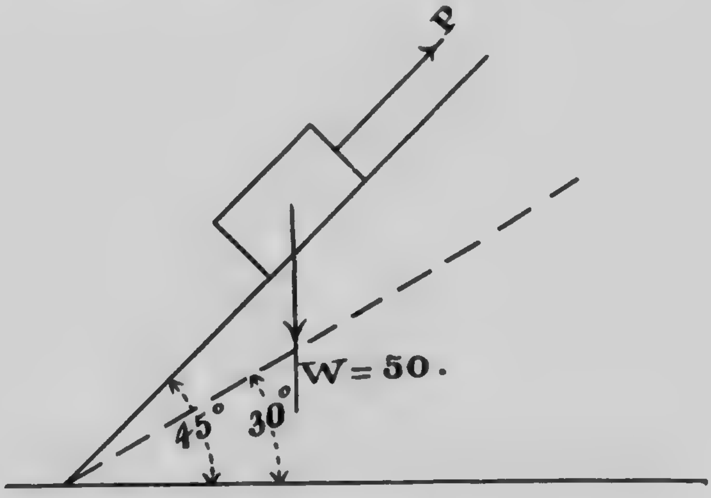


Fig. 56.

Fig. 56 represents the condition of affairs.

$$\Sigma X = X_W + X_P + X_F = 0.$$
$$-50 \cos 45^\circ + P - F = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (7.)$$

$$F = N.\mu.$$
$$= 50 \cos 45^\circ . \frac{1}{\sqrt{3}} = 50 . \frac{1}{\sqrt{2}} . \frac{1}{\sqrt{3}}.$$

Substituting this value of F into (7.),

$$-50\frac{1}{\sqrt{2}} + P - \frac{50}{\sqrt{2}}\cdot\frac{1}{\sqrt{3}} = 0.$$

$$P = \frac{50}{\sqrt{2}}\left(1 + \frac{1}{\sqrt{3}}\right).$$

CHAPTER XIII.

WIND PRESSURE.

Pressure on Surfaces Normal to Direction of Motion.

It will be primarily assumed that, for any stated velocity of wind, the motion of the air particles is uniform for all positions on any given cross section, no account being taken of interference of particles with one another or of eddies.

All calculations will be made at 60° Fahr. temperature and 760 cms. barometric pressure. Under these conditions one cubic foot of air weighs .078 pounds.

Let the velocity of the wind be "V" feet per second. Consider any cross section of one square foot area. The volume of air that will flow past this section in **one second** is evidently:—

$$V \times 1 \text{ cubic feet} \ldots\ldots\ldots\ldots\ldots\ldots \quad (1.)$$

or, in other words, since the flow is steady (uniform), V cubic feet of air flow past this section per second. Multiplying (1.) by .078 gives the flow in pounds per second as:—

$$V \times 1 \times .078 = .078\,V \text{ pounds per second} \ldots \quad (2.)$$

Let dW be the weight of air flowing past this same section in a time "dT"; then, evidently,

$$dW = (.078\,V)\,dT.$$

Now, let attention be confined to an area of one square foot of any surface normal to the motion of the air particles. Considering the mass of air as being made up of a number of small masses of magnitude "dM," the action of each of these small particles as it strikes the normal surface is of an impulsive nature of duration "dT," such that:—

$$P.dT = dM.V. \ldots\ldots\ldots\ldots\ldots\ldots \quad (3.)$$

$$\text{but } dM = \frac{dW}{g}$$

$$= (.078\,V)\,dT/32.2.$$

Substituting this value of dM in (3.), it is seen that $P.dT = (.078\,V)\,dT.V/32.2$;

or,

$$P = .078\,V^2/32.2. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

where P is the impulsive force acting on the normal surface; and, since the impulse is distributed over one square foot area, P may be said to be the resultant pressure per square foot.

In equation (4.) V is in feet per second, but it is usual to express wind velocities in miles per hour. Let V′ be the wind velocity in miles per hour; then

$$\begin{aligned} V &= V' \times 5280/60 \times 60. \\ &= 1.47\,V'. \end{aligned}$$

Substituting this value of V in (4.), we see that

$$\begin{aligned} P &= .078\,(1.47\,V')^2/32.2. \\ &= .0052\,V'^2, \end{aligned}$$

which result, as would be reasonably expected, is higher than practical experiments show.

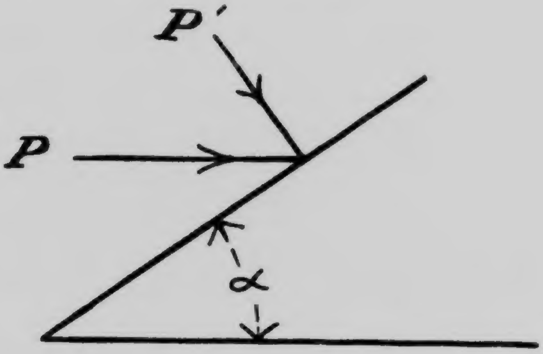


Fig. 57.

Pressure on Inclined Surfaces.

Let P (Fig. 57) represent the resultant wind pressure on one square foot area, such that

$$P = .0052\,V^2,$$

where V = velocity of wind in miles per hour.

Resolve P perpendicular and parallel to the plane surface, such that P′ is the normal resolved part. If α be the inclination of the plane to the direction of flow of wind, which is generally considered as being horizontal, then

$$\begin{aligned} P' &= P \sin \alpha. \\ &= .0052\, V^2 \sin \alpha. \end{aligned}$$

P′ is the effective wind pressure, and is, of course, less than P.

If A be the total area of any surface in square feet, then the resultant wind pressure is

$$P'' = P'.A.$$